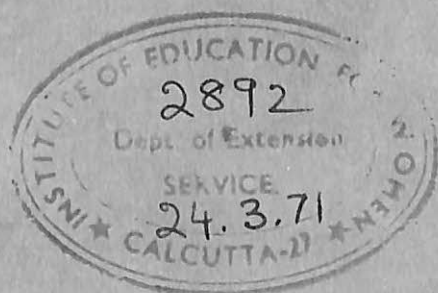


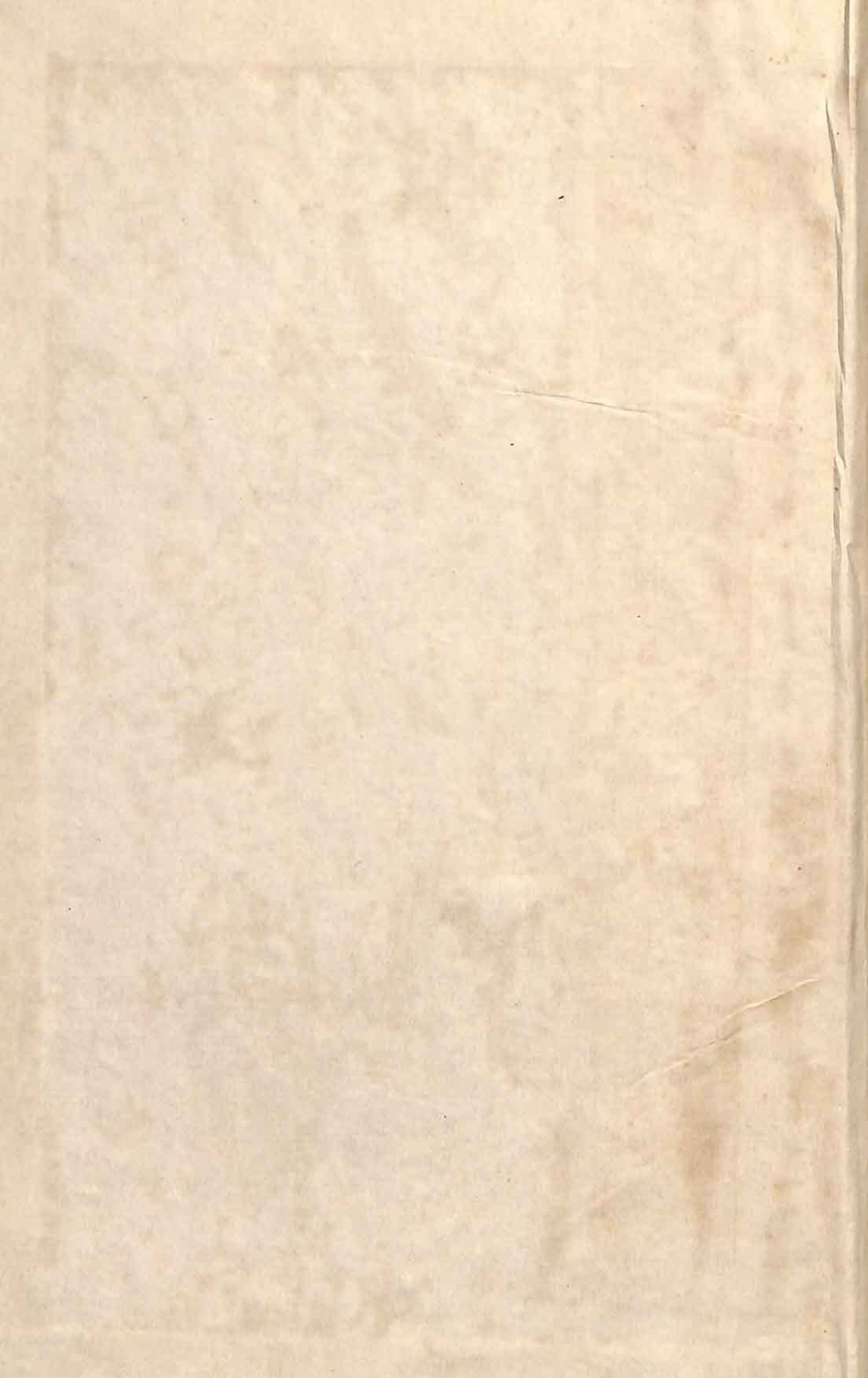
ELEMENTS OF MATHEMATICAL ANALYSIS

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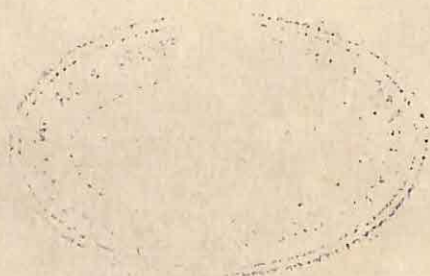
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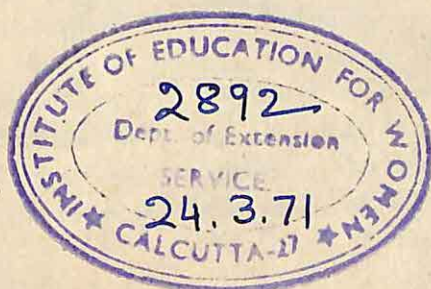
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ELEMENTS OF MATHEMATICAL ANALYSIS

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P R E F A C E

Mathematical Analysis now occupies a prominent place in the curriculum of the Degree Examinations in Mathematics in many Indian universities. But unfortunately there is no elementary book on the subject suitable for under-graduate students. The books that are available, are either of a higher standard and meant for post-graduate and research students or else are incomplete in the sense that they do not cover the entire syllabus. The present venture is an attempt to place in the hands of the under-graduate students a really elementary book on the subject which they can read with pleasure and profit. In preparing the book, the author had to consult the standard works on the subject and he takes this opportunity to express his gratefulness to their authors.

An interesting feature of the book is that it contains a large collection of examples selected from Calcutta University Papers and other sources. Some of these are worked out in details ; hints are given for the solution of a good many of them. It is expected that students will be able to follow the book and work out the examples without any external help.

It gives me great pleasure to express my cordial thanks to Shri Bibhuti Bhusan Ghose of Kalpana Prokashani for his unfailing interest in the publication of the book. Printing of a mathematical book containing a large number of symbols of various types is always a difficult job. Thanks to the members of the Ramkrishna Press for the way in which they have done their work.

It is not unlikely that there are errors in the book and the author will remain grateful to them who will please point them out.

Calcutta
15th August, 1967.

G. D. B

SECOND EDITION

We offer our heartfelt thanks to the Professors for their kind appreciation of the First edition of the book, which has encouraged us in bringing out the Second edition.

We had a mind to make improvements in the present edition but the sad and sudden death of Professor G. D. Bhar has impaired all our efforts. Only a few of the valuable suggestions kindly made by some professors have been incorporated in the book in this edition. In fact, it is almost a reprint of the previous edition with some additions in the Chapter on "The Number System" only. In this connection we take the opportunity of expressing our gratitude to Prof. K. Chakraborty of Chandernagore College who has kindly given us suggestions for its improvements, which along with the same received from others, we expect to incorporate in the next edition of the book.

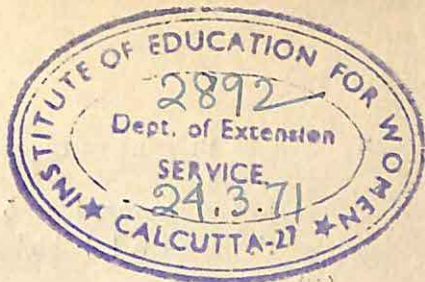
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CHAPTER 1

THE MODULUS NOTATION

1.1. The modulus of a real* number.

Definition—The modulus or the absolute value of a real number a , is denoted by the symbol $|a|$, and is equal to a , 0, or $-a$ according as a is positive, zero, or negative. Thus

$$|a| = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

The modulus of a real number is never negative. Also we have for every a

$$a \leq |a|, \text{ also } -a \leq |a|. \quad \dots \quad (1)$$

1.2. Some theorems involving moduli.

Theorem I. $|a+b| \leq |a| + |b|$ (2)

(i) Let $a+b$ be positive. We have by (1)

$$|a| + |b| \geq a+b,$$

and $a+b = |a+b|$, by definition.

$$\therefore |a| + |b| \geq |a+b|.$$

(ii) Let $a+b$ be negative. We have by (1),

$$|a| + |b| \geq (-a) + (-b),$$

and $(-a) + (-b) = -(a+b) = |a+b|$, by definition.

$$\therefore |a| + |b| \geq |a+b|.$$

(iii) Let $a+b$ be zero. We have

$$|a| + |b| \geq a+b, \text{ by (1),}$$

and $a+b = |a+b|$ for each $= 0$.

$$\therefore |a| + |b| \geq 0, \\ \geq |a+b|.$$

Hence in every case

$$|a+b| \leq |a| + |b|.$$

Theorem II. $|a-b| \leq |a| + |b|$ (3)

(i) Let $a-b$ be positive.

$$|a| + |b| \geq (a) + (-b), \text{ by (1),}$$

* For the definition of a real number see Chap. 2.

$$\text{and } a-b = |a-b|$$

$$\therefore |a| + |b| \geq |a-b|.$$

(ii) Let $a-b$ be negative.

$$|a| + |b| \geq (-a) + (b)$$

$$\text{and } (-a) + (b) = -(a-b) = |a-b|.$$

$$\therefore |a| + |b| \geq |a-b|.$$

(iii) Let $a-b$ be zero,

$$|a| + |b| \geq (a) + (-b)$$

$$\text{and } a + (-b) = a-b = 0 = |a-b|.$$

$$\therefore |a| + |b| \geq |a-b|.$$

Hence in every case

$$|a-b| \leq |a| + |b|$$

Combining theorems I and II, we may write

$$|a \pm b| \leq |a| + |b|.$$

Theorem III. $|a \pm b| \geq |a| - |b|.$

We have $a = a+b-b$.

$\therefore$ By (3),

$$|a| = |(a+b)-b| \leq |a+b| + |b|$$

$$\text{By transposition, } |a+b| \geq |a| - |b| \quad \dots \quad (4)$$

Again, $a = a-b+b$;

$$\therefore |a| = |(a-b)+b| \leq |a-b| + |b|, \quad [\text{by (2)}]$$

$$\text{By transposition, } |a-b| \geq |a| - |b|. \quad \dots \quad (5)$$

Combining (4) and (5), we get

$$|a \pm b| \geq |a| - |b|. \quad \dots \quad (6)$$

Theorem IV. $|a-b| = |b-a|.$

(i) Let $a-b > 0$, then $b-a < 0$,

$$\therefore |b-a| = -(b-a) = (a-b) = |a-b|.$$

(ii) Let $a-b < 0$, then $b-a > 0$.

$$\therefore |a-b| = -(a-b) = b-a = |b-a|.$$

(iii) If $a-b=0$, then $b-a=0$,

$$\therefore |a-b| = 0 = |b-a|.$$

Theorem V. To show that $|a-b| < c$ is equivalent to

$$(i) \quad b-c < a < b+c,$$

$$(ii) \quad -c < a-b < c.$$

$$\text{If } a-b > 0, |a-b| < c \text{ is } a-b < c, \text{ or } a < b+c; \dots \quad (7)$$

If $a-b < 0$, $|a-b| < c$ is $-(a-b) < c$, or $a-b > -c$;
i.e., $a > b-c$ (8)

Combining (7) and (8), we have

$$b-c < a < b+c.$$

By transposition, $-c < a-b < c$.

Cor. 1. $|a-b| \leq c$ is equivalent to $b-c \leq a \leq b+c$,
 or, $-c \leq a-b \leq c$.

Cor. 2. $|x| < a$ is equivalent to $-a < x < a$.

Thus the *open* interval $(-a, a)$ is $|x| < a$.

Cor. 3. $|x| \leq a$ is equivalent to $-a \leq x \leq a$.

Thus the *closed* interval $(-a, a)$ is $|x| \leq a$.

Examples

1. If $|a-b| < A$ and $|b-c| < B$, prove that
 $|a-c| \leq A+B$.

2. Find the value of $\frac{a+|a|}{2}$ when (i) $a > 0$ and (ii) $a < 0$.

3. Express in modulus notation the relation satisfied by points x at a distance less than or equal to 3 units from the point 3 on the right and left of it.

4. What intervals are represented by :

(a) $|x-2| < 2$; (b) $|2-x| \leq 3$; (c) $|x-a| \leq b$?

5. Express in modulus notation the open interval $(2, 3)$ and the closed intervals (a, b) , $(-a, -b)$.

6. Prove that

(i) $|a_1+a_2+a_3| \leq |a_1| + |a_2| + |a_3|$;

(ii) $|a_1+a_2+\dots+a_n| \leq |a_1| + |a_2| + \dots + |a_n|$.

7. Prove that

(i) $|a \cdot b| = |a| \cdot |b|$;

(ii) $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$, if $b \neq 0$.

[Ans. 2. (i) a ; (ii) 0. 3. $|x-3| \leq 3$. 4. (a) open interval $(0, 4)$; (b) closed interval $(-1, 5)$; (c) closed interval $(a-b, a+b)$.

5. $\left| x - \frac{5}{2} \right| < \frac{1}{2}$; $\left| x - \frac{a+b}{2} \right| \leq \frac{b-a}{2}$; $\left| x + \frac{a+b}{2} \right| \leq \frac{a-b}{2}$].

CHAPTER 2

THE NUMBER SYSTEM

2.1. Introductory.

The whole structure of Mathematical Analysis is based upon the system of real numbers. We begin with a study of this system.

The earliest concept of numbers originated from counting and the first set of numbers known was the set of positive integers or 'natural numbers'. Others had to be introduced as the necessity for them arose by a series of generalisations.

With positive integers the operation of subtraction is not always possible, and the negative integers had to be introduced. Then again the operation of division is not always possible with the integers, positive and negative, unless we introduce the fractional numbers, or, as they are called, the rational numbers. In other words, we cannot solve equations of the type $x+a=0$, where a is a positive integer unless we introduce the negative integers. But even with all the integers, positive and negative, we cannot solve the equation $ax+b=0$, in which a and b are integers prime to each other, unless we have the fractional numbers or the rational numbers.

2.2. Rational Numbers.

If $r = \frac{m}{n}$, where m and n are integers, r is called a *rational number*. We may assume that (i) m and n are prime to each other, and (ii) n is positive. If $m=0$, we get the rational number zero. It may be noted here that all integers may be looked upon as rational numbers for which $n=1$.

The totality of numbers, positive and negative integers and positive and negative fractions, constitutes the aggregate of rational numbers.

2.3. Fundamental properties of the system of rational numbers.

A. Relations of order among rational numbers.

If a, b, c are members of the system of rational numbers, then

1. Either $a > b$, $a = b$, or $a < b$. Law of trichotomy.

2. If $a > b$ and $b > c$, then $a > c$. Law of transitivity.

B. Operations with rational numbers.

If a , b , c are members of the system of rational numbers, then

1. $a+b$ and ab belong to the system. Closure law.
2. $a+b=b+a$. Commutative law of addition.
3. $a+(b+c)=(a+b)+c$. Associative law of addition.
4. $ab=ba$. Commutative law of multiplication.
5. $a(bc)=(ab)c$. Associative law of multiplication.
6. $a(b+c)=ab+ac$. Distributive law.
7. $a+0=0+a=a$;
 $a \cdot 1=1 \cdot a=a$.

0 is called the *identity with respect to addition*; 1 is called the *identity with respect to multiplication*.

8. For any a there is a number x in the system such that $x+a=0$. x is called the *inverse* of a with respect to addition or the *negative* of a and is denoted by $-a$.

9. For $a \neq 0$, there is a number x in the system such that $ax=1$. x is called the *inverse* of a with respect to multiplication or the *reciprocal* of a and is denoted by a^{-1} or $\frac{1}{a}$.

These enable us to operate with the rational numbers according to the usual rules of algebra. A system whose members satisfy the above is called a *field*.

C. Inequalities.

1. If $a > b$, then $a+c > b+c$.
2. If $a > b$ and $c > 0$, then $ac > bc$.
3. If $a > b$ and $c < 0$, then $ac < bc$.

D. Axiom of Archimedes.

If a and b are positive rational numbers such that $0 < a < b$, we can always find a positive integer n so that $na > b$ and consequently $\frac{b}{n} < a$.

E. Between any two unequal rational numbers, however nearly equal, we can insert as many other rational numbers as we please. If a and b ($a < b$) be rational numbers, however nearly equal, the set of rational numbers

$$a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, a + \frac{(n-1)(b-a)}{n}, b$$

is obviously in order of magnitude and the integer n may be chosen as large as we please. We express this by saying that the system of rational numbers is a *compact* system or that the system is *dense*.

From the above it follows that given any rational number, we can find rational numbers as close to it as we please, both on its right and on its left.

2'4. Geometrical representation of rational numbers.

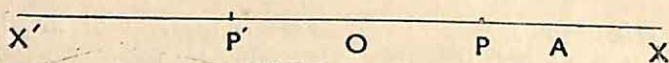


Fig. 1

Take a straight line $X'X$ of indefinite length as axis of reference and a fixed point O on it as origin. A suitable length OA on it being chosen as unity, if we divide OA into n equal parts, and take a length OP (or OP') equal to m (m may be $< n$, Fig 1, or $> n$) such parts (P being to the right of O , if m be positive and P' being to the left of O , if m be negative), the point P (or P') represents the rational number $\frac{m}{n}$. The point P representing a rational number is sometimes called a *rational point*.

As the aggregate of rational numbers is everywhere dense, if P be a rational point, we can find another rational point Q as close to it as we please. It would appear, therefore, that the line $X'X$ is made up of rational points only. But this is not so. In other words, whereas all the rational numbers are represented by points on the axis and though in any segment of the axis however small we can find as many rational points as we please, still the converse that every point on the axis must represent some rational point is not true. We shall see presently that the axis is incalculably richer in points than the aggregate of rational numbers.

2'5. Irrational Numbers.

A further necessity for extending the idea of numbers arises when we consider equations of the type $x^2 - 2 = 0$; for with all the rational numbers at our disposal, we cannot solve this equation, as there is no rational number whose square is 2.

If possible, let m and n be two positive integers, prime to each other, such that

$$\left(\frac{m}{n}\right)^2 = 2, \text{ i.e., } m^2 = 2n^2.$$

Now since m^2 is clearly even, so must m be. Let $m = 2m'$, where m' is an integer. We then have,

$$4m'^2 = 2n^2, \text{ or, } n^2 = 2m'^2$$

and therefore n must be even. Thus m and n have a common factor 2 which contradicts the hypothesis that m and n are prime to each other. Therefore, there is no rational number whose square is 2.

This may be geometrically interpreted as follows. OA being unity, if AB is drawn perpendicular to OA and equal to OA and OB be joined, it follows from elementary geometry that $OB^2 = 2$. If we now measure off

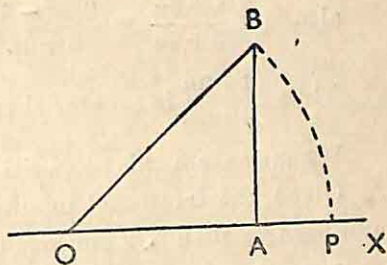


Fig. 2

a length OP from the axis OX , equal to OB , the point P represents a number whose square is 2, and we have seen that this is not a rational number. Thus, there are points on the axis which are not rational points.

To solve equations of the type $x^2 - 2 = 0$, or viewed geometrically, to take into account points like P lying on the axis, we have to extend our idea of numbers and introduce what are called *irrational numbers*.

Let $\sqrt{2}$ denote the number whose square is 2 and let P be the point on the axis which represents it; we shall call $\sqrt{2}$ an *irrational number* and the point P an *irrational point*.

2.6. Given an irrational number, we can find rational numbers as close to it as we please.

Let $\sqrt{2}$ be the given irrational number and let m be a positive rational number such that $m^2 < 2$.

Consider now the positive rational number $\frac{4+3m}{3+2m}$. We have

$$2 - \left(\frac{4+3m}{3+2m}\right)^2 = \frac{2-m^2}{(3+2m)^2} > 0, \text{ i.e., } \left(\frac{4+3m}{3+2m}\right)^2 < 2.$$

$$\text{Again } \frac{4+3m}{3+2m} - m = \frac{2(2-m^2)}{3+2m} > 0, \text{ so that } \frac{4+3m}{3+2m} > m.$$

Thus $\frac{4+3m}{3+2m}$ is less than $\sqrt{2}$ and closer to it than m .

Next let n be a positive rational number such that $n^2 > 2$ and consider the positive rational number $\frac{4+3n}{3+2n}$. We have

$$\left(\frac{4+3n}{3+2n}\right)^2 - 2 = \frac{n^2 - 2}{(3+2n)^2} > 0, \text{ i.e., } \left(\frac{4+3n}{3+2n}\right)^2 > 2;$$

$$\text{also } n - \frac{4+3n}{3+2n} = \frac{2(n^2 - 2)}{3+2n} > 0, \text{ or, } n > \frac{4+3n}{3+2n}.$$

Thus $\frac{4+3n}{3+2n}$ is greater than $\sqrt{2}$ and closer to it than n .

We may state this proposition as follows :

Given the irrational number $\sqrt{2}$, there is no greatest rational number less than $\sqrt{2}$ nor any least rational number greater than $\sqrt{2}$.

2.7. Sections of Rational Numbers.

Suppose by some means or other we divide the set of rational numbers into two classes, L and R , in such a way that

(i) each class exists, so that all the rational numbers do not belong to the same class making the other class empty ;

(ii) each number has a class, i.e., no number escapes classification ;

(iii) every member of L is less than every member of R .

A division of the set of rational numbers in this manner into two classes L and R is called a *section* of the rational numbers and is denoted by (L, R) ; L is called the *lower* class and R the *upper* class of the section.

It is easy to show that if (L, R) is a section, any rational number less than a member of L is also a member of L and any rational number greater than a member of R is also a member of R .

2.8. Illustrations of three types of sections.

Suppose the rational numbers are represented by points on a line of indefinite length and let the line be cut at any point P .



Fig. 3

The system of rational numbers is thereby divided into two classes, those lying to the left of P forming the lower class L and

those lying to the right of P forming the upper class R . If P is rational, it belongs to L or to R . Both classes exist; every rational number belongs to one or the other of the two classes and any number in L is less than any number in R . So this division of the rational numbers constitutes a section as defined above. Three cases arise:

First case: Suppose P is a rational point representing the number 5 for example, and suppose L contains all rational numbers ≤ 5 ; R then contains all rational numbers > 5 . Thus L has a greatest number 5, but R has no least number.

Second case: Again suppose that P represents the rational number 5 and that L contains all rational numbers < 5 and R contains 5 and all rational numbers > 5 . In this case L has no greatest number, but R has a least number, namely 5.

Third case: Suppose that P is not a rational point and let it represent the number whose square is 2. In this case L contains all negative rational numbers, zero, and all positive rational numbers whose squares are less than 2 and R contains all positive rational numbers whose squares are greater than 2. As the number whose square is 2 is not a rational number (already proved), no rational number escapes classification. Also L has no greatest number and R no least number (Art. 2'6).

Note. There cannot be a fourth case in which L has a greatest number and R a least number. For, if a be the greatest number of L and b the least number of R , the infinite number of rational numbers lying between a and b would escape classification.

The first two cases are said to define the rational number 5, and 5, which is the greatest number of L in the first case and the least number of R in the second case, is sometimes called the 'frontier' number. In the third case, where L has no greatest number and R has no least number, we assume the existence of a number whose square is 2. We call this an *irrational* number and represent it by the symbol $\sqrt{2}$. It separates the L -class from the R -class and we can find rational numbers as close to it as we please, both in the L -class and in the R -class.

The aggregate of numbers, rational and irrational, is called the system of *real numbers*.

2'9. Dedekind's definition of real numbers.

Dedekind has employed the method of sections to define real numbers.

If by some means or other we divide all rational numbers into two classes L and R such that

- (i) each class contains at least one rational number,
- (ii) every rational number belongs to one or other of the two classes,

and (iii) each number of L is less than every number of R , then we get a section of the rational numbers which defines a number, rational or irrational, the particular way of division defining a particular number by its section.

Three cases arise :

- (1) The L -class has a greatest number, but the R -class has no least number.
- (2) The L -class has no greatest number but the R -class has a least number.
- (3) The L -class has no greatest number and the R -class has no least number.

It is not possible to have a case in which the L -class has a greatest number and at the same time the R -class has a least number.

Such divisions of the system of rational numbers into mutually exclusive classes defining numbers are called Dedekind Sections. Sections corresponding to the first two cases define rational numbers whereas the section corresponding to the third case defines an irrational number.

This extension of our conception of numbers as sections of the system of rational numbers gives us a more satisfactory basis of defining numbers in a uniform manner. We substitute sections of rational numbers for the numbers defined by them and are led to a generalisation of our number system. The aggregate of sections of rational numbers is greater than the aggregate of rational numbers, as there are sections not corresponding to any rational number. The totality of numbers defined as sections of rational numbers is the aggregate of *real* numbers.

2.10. Modified definition of a section.

From the above it is clear that to each rational number there correspond two sections, one in which it is the greatest number of the lower class and the other in which it is the least number of the upper class. In order to have a unique section for each number,

rational, or irrational, we modify the definition of a section by stipulating that *the lower class must not have a greatest number*. We would then have the following two kinds of sections.

(1) A section in which L has no greatest number, but R has a least number. This section corresponds to a rational number.

(2) A section in which L has no greatest number and R has no least number. It defines an irrational number.

Our modified definition of a section is therefore :

Any division of the system of rational numbers into two classes is a section, if (i) each class exists, (ii) every rational number has a class, (iii) every member of L is less than every member of R , and (iv) L has no greatest number.

The frontier number, if it exists, is thus always in the R -class.

The number defined by the section will be denoted by the symbol (L, R) .

It is easily proved that any number less than a number of L is a member of L and any number greater than a number of R is a member of R .

2'11. Real Rational Number ; Real Irrational Number.

Since a section of rational numbers is a real number, which may be rational or irrational, a section which corresponds to a rational number m is called a *real rational number* ; if (L, R) be the section which defines it, m is the least number of R . To every rational number expressed as the ratio of two integers there corresponds a section which defines it, so that there is a one-to-one correspondence between rational numbers and real rational numbers. There is a conceptual distinction between a rational number expressed as a ratio and the section which defines it, that is, between a rational number and a real rational number. The real rational number corresponding to the rational number m , will be denoted by $\bar{m}$.

An irrational number, however, would always mean a *real irrational number* as it has not been otherwise defined.

2'12. Definitions of some real numbers.

1. The real number 0.

It is defined by the section (L, R) in which all the negative

rational numbers are put in L ; 0 and all positive rational numbers in R .

2. A positive real number.

The section $\alpha = (L, R)$ in which L contains some positive rational numbers defines a *positive* real number α .

3. A negative real number.

The section $\alpha = (L, R)$ in which R contains some negative rational numbers defines a *negative* real number α .

4. The negative of a real number α .

If $\alpha = (L, R)$, the negative of α , written $-\alpha$, is defined by the section (L', R') where L' contains the negatives of all numbers in R and R' contains the negatives of all numbers in L , with the restriction that if a frontier number exists for the section (L, R) , its negative is to be put in R' .

5. The reciprocal of a non-zero real number α .

If $\alpha = (L, R)$, the section (L', R') where L' contains the reciprocals of the numbers in R , and R' contains the reciprocals of the numbers in L , the reciprocal of the frontier number in (L, R) , if it exists, being put in R' , defines the reciprocal of α .

2.13. Relations of order. Algebraic operations with real numbers defined as sections.

Numbers defined as sections cannot be accepted as numbers until we establish relations of order among them and show that they obey the laws of algebraic operations such as addition, subtraction, multiplication, division, etc.

1. Relations of order.

If $\alpha = (L_1, R_1)$ and $\beta = (L_2, R_2)$,

(i) $\alpha = \beta$, if L_1 and L_2 are identical.

(ii) $\alpha > \beta$, if L_2 is a part of L_1 , but L_1 contains numbers which do not occur in L_2 .

(iii) $\alpha < \beta$, if L_1 is a part of L_2 , L_2 having numbers not occurring in L_1 .

Cor. From these definitions it is easy to prove that if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.

We make use of the following theorem whenever required in the sections that follows :—

Theorem : *Any given set of rational numbers can constitute the L -class of a section, if and only if, (i) all the rational numbers do not belong to the set ; (ii) it has no greatest number ; and (iii) a rational number less than any number of the set is also in the set.*

2. Addition.

Given two real numbers $\alpha=(L_1, R_1)$ and $\beta=(L_2, R_2)$ we have to find a section $\gamma=(L, R)$ to be called the sum of α and β , i.e., $\gamma=\alpha+\beta$.

To do this we form a class L by putting in it rational numbers obtained by adding every number of L_1 to every number of L_2 . Clearly L exists and does not contain all the rational numbers. In order that L can be the lower class of a section, we have to show that (i) L has no greatest number and (ii) a number less than any number of L is also a member of L .

To prove (i), let us suppose that the number a where $a=a_1+a_2$, a_1 being a number of L_1 and a_2 of L_2 , is the greatest number of L . As L_1 has no greatest number, we can find a number a_1+x of L_1 , greater than a_1 , where x is a positive rational number. Therefore, a_1+x+a_2 or $a+x$ is a number of L ; and this being greater than a , a cannot be the greatest number of L . Thus L has no greatest number.

To prove (ii), let a' be a number less than any number a of L , where $a=a_1+a_2$, a_1 being a number of L_1 and a_2 of L_2 . Let $a'=a-x$, x being a positive rational number. $\therefore a'=a-x=a_1+a_2-x=(a_1-x)+a_2$, and a_1-x being less than a_1 is a number of L_1 and a_2 of L_2 ; so $(a_1-x)+a_2$ or $a-x=a'$ is a member of L .

It is thus proved that L can be the lower class of a section, the upper class R being formed of all the rational numbers that do not occur in L . We thus have $\gamma=\alpha+\beta$, or $(L, R)=(L_1, R_1)+(L_2, R_2)$.

3. Subtraction.

$\alpha-\beta$ is defined by $\alpha+(-\beta)$ and is thus a corollary to addition.

4. Multiplication.

To define the product $\alpha\beta$, we first consider $\alpha=(L_1, R_1)$ and $\beta=(L_2, R_2)$ to be positive.

We form a class L consisting of all negative rationals, zero, and all those positive rationals which are obtained by multiplying every positive number of L_1 with every positive number of L_2 .

Clearly L exists and does not contain all the rational numbers.

L has no greatest number. If possible, let $a = a_1 a_2$, obtained as the product of a_1 of L_1 and a_2 of L_2 , be the greatest number of L . Since L_1 has no greatest number, we can find a number b_1 of L_1 such that $b_1 > a_1$. Let $\frac{b_1}{a_1} = x$, a positive rational number > 1 ; then $b_1 = a_1 x$. Now the product $b_1 \cdot a_2 = (a_1 x) \cdot a_2 = a_1 a_2 x = ax > a$, and it belongs to L . So a is not the greatest number of L . In other words, L has no greatest number.

Next let a' be a positive rational number smaller than a positive number a of L and $a = a_1 a_2$ where a_1, a_2 are positive numbers of L_1, L_2 respectively. Let $\frac{a'}{a} = x$, so that $0 < x < 1$; we now have $a' = ax = a_1 a_2 x = (a_1 x) \cdot a_2$. The number $a_1 x (< a_1)$ belongs to L_1 and a_2 to L_2 . Thus $a' (< a)$, being the product of a positive number $a_1 x$ of L_1 and a positive number a_2 of L_2 , belongs to L .

Thus L can be the lower class of a section (L, R) and we have $\alpha, \beta = (L_1, R_1) (L_2, R_2) = (L, R)$.

We extend this definition to negative numbers by means of the relations :

$$\alpha, (-\beta) = -(\alpha, \beta); (-\alpha), \beta = -(\alpha, \beta); (-\alpha), (-\beta) = \alpha, \beta.$$

5. Division.

The division of α by β , where $\beta \neq 0$, is defined by the product $\alpha, \frac{1}{\beta}$.

2.14. Properties of Real Numbers.

(i) All the fundamental laws of algebra which hold for rational numbers can be proved to hold for the real numbers.

(ii) The real numbers are ordered.

(iii) They are dense.

We prove here two properties that will be required hereafter.

(i) If a and b be any two rational numbers belonging to L and

R respectively of the real number $\alpha \equiv (L, R)$, then $\bar{a}$, the real rational number, as also $\bar{b}$, are such that

$$\bar{a} < \alpha \leq \bar{b}.$$

Proof : Let $\bar{a} \equiv (L_1, R_1)$ and $\bar{b} \equiv (L_2, R_2)$; then a is the least number of R_1 and b is that of R_2 . Hence L_1 contains all rational numbers $< a$. Now a belongs to L , therefore all rational numbers $< a$ also belong to L . Again a is of L but not of L_1 , therefore L_1 is contained in L . Hence by the relation order of real numbers $\bar{a} < \alpha$.

Similarly we can show that L is contained in L_2 , so that $\alpha \leq \bar{b}$, the equality sign coming when b of R is the least member of R .

(ii) Between any two different real numbers there is an infinite number of real rational numbers.

Proof : Let the different real numbers be $\alpha \equiv (L_1, R_1)$ and $\beta \equiv (L_2, R_2)$ and let $\alpha < \beta$. Then L_1 is a part of L_2 . Therefore, there is an infinity of rational numbers of L_2 not belonging to L_1 and hence belonging to R_1 . If a be any one of those, then $\alpha < \bar{a}$, the real rational number corresponding to a . Again a belongs to L_2 , therefore $\bar{a} < \beta$. Thus $\alpha < \bar{a} < \beta$. Hence the theorem.

215. Sections of Real Numbers. Dedekind's Theorem.

We have seen that by dividing the system of rational numbers in sections, we get rational as well as irrational numbers. We are thus led to an extension of the number system by taking cognisance of the irrational numbers. It may be supposed that by dividing the system of real numbers in sections, we may get a further extension of our concept of numbers. But this is not so.

In this connection we state the following theorem, known as Dedekind's theorem on sections of real numbers; and for the proof we make use of § 29 for the definition of real numbers.

Dedekind's Theorem.

"If real numbers be divided into two classes L and R , in such a way that

- (i) every number belongs to one or other of the two classes,
- (ii) each class contains at least one number,

(iii) any member of L is less than any member of R , then there is a number α , which has the property that all numbers less than it belong to L and all numbers greater than it to R . The number α itself may belong to either class."

Proof: Let L_1 and R_1 be the two classes consisting of the rational numbers corresponding to the real rational numbers of L and R respectively.

Then (L_1, R_1) being now a section of rational numbers, the following cases may arise :—

First Case: L_1 contains the greatest rational number a , say. Then the real rational number $\bar{a}$ of L will be the greatest real number of L .

If not, let α of $L > \bar{a}$.

Now between the two real numbers of L , viz., $\bar{a}$ and α there exist infinite number of real rational numbers. Let $\bar{b}$ be one of them.

Then $\bar{a} < \bar{b} < \alpha$.

Again $\bar{b}$ being $< \alpha$ of L must belong to L and hence b the corresponding rational number must belong to L_1 . But in L , $\bar{a} < \bar{b}$, therefore in L_1 , $a < b$.

Thus a is not the greatest member of L_1 . This contradicts our assumption. Therefore $\bar{a}$ of L is the greatest member of L .

Second Case: R_1 has the least number c , say. Then $\bar{c}$ can be shown as above to be the least member of R .

Third Case: L_1 may not have the greatest member and R_1 not the least member.

In this case section (L_1, R_1) corresponds to an irrational number α , say, which may belong either to L or to R .

Let α belong to L , then α is the greatest member of L . If not, let β of L be $> \alpha$.

α and β being two different real numbers, an infinity of real rational numbers lie between them; let $\bar{d}$ be one of them,

$$\text{i.e., } \alpha < \bar{d} < \beta.$$

$\bar{d}$ being $< \beta$ of L must belong to L . So d , the corresponding rational number, belongs to L_1 .

Hence we have found a rational number d of L_1 , of the section $\alpha \equiv (L_1, R_1)$ such that $\alpha < \bar{d}$, which is absurd. Therefore α is the greatest member of L .

Similarly if α belongs to R , then it can be proved as above that α is the least number of R .

Thus, in all the possible cases we get a real number corresponding to the section (L, R) of real numbers. Hence the theorem of Dedekind is established.

Thus, as sections of real numbers we get real numbers only and not any other type of numbers. No further extension of the idea of numbers is possible by this method.

2'16, Cantor-Dedekind axiom. The continuum.

The system of real numbers having no gaps is said to form a continuum—the arithmetic continuum.

When we come to the representation of points on a line, we have seen that the rational numbers can be so represented on a line, which we may call the number-axis. Also for some irrational numbers like $\sqrt{2}$, we have seen (Art. 2'5) that there is a point on the number-axis corresponding to it. But we cannot say this for every irrational number. We assume that there is one and only one point on the number-axis corresponding to every real number. This is known as Cantor-Dedekind axiom. By means of this axiom a correspondence is established between the aggregate of real numbers—the arithmetic continuum—with the aggregate of points on a line—called the linear continuum.

The aggregate of real numbers or the continuum has a kind of completeness which the aggregate of rational numbers lacked—a completeness which is expressed by saying that the continuum is closed.

Examples.

1. If m and n are rational numbers, show that $m+n$, $m-n$, $m \times n$ and m/n are rational numbers where in the last case $n \neq 0$.

2. If l , m , n are positive rational numbers and $m > n$, then $l(m^2 - n^2)$, $2lmn$, $l(m^2 + n^2)$ are positive rational numbers. Hence show how to determine any number of right-angled triangles the lengths of all of whose sides are rational. (Hardy).

3. The positive rational numbers may be arranged in the form of a series as follows :

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \dots$$

Show that p/q is the

$$\left[\frac{1}{2}(p+q-1)(p+q-2)+q \right] \text{th}$$

term of the series. (*Hardy*)

4. Prove that no rational number can have its cube equal to 2.

[*Solution.* Since $1^3=1$ and $2^3=8$, if there be a rational number $\frac{p}{q}$ in its lowest terms such that $\left(\frac{p}{q}\right)^3=2$, then $1 < \frac{p}{q} < 2$,

so that $q \neq 1$. Now $\frac{p^3}{q^3}=2$ gives $p^3=2q^3$, or $\frac{p^3}{q}=2q^2$. q being

prime to p , it is prime to p^3 and as $q \neq 1$, $\frac{p^3}{q}$ is a fraction whereas

$2q^2$ is an integer. Thus we find that a fraction is equal to an integer, which is absurd. Thus, there is no rational fraction whose cube is 2.]

5. A rational number p/q in its lowest terms cannot be the square of another rational number unless p and q are both perfect squares.

[For solution see *Hardy—Pure Mathematics*, 10th edition, page 6 ; or Ex. 21 below.]

6. Show that an irrational number is never equal to a rational number.

7. Show that between two different real numbers α and α' there is an infinite number of rational numbers. Consider separately the cases ; (i) α and α' are both rational ; (ii) one of α and α' is rational, the other is irrational ; (iii) both α and α' are irrational.

8. Prove that between two different real numbers α and α' there is an infinite number of irrational numbers.

[*Solution.* Suppose $\alpha < \alpha'$. We can always find two rational numbers β and β' between α and α' such that $\alpha < \beta < \beta' < \alpha'$. We proceed to show that there is an irrational number γ between β and β' . Given any irrational number γ , we can always find two rational numbers, m and n ($m < n$) such that $m < \gamma < n$ and $n-m$ is as small as we please and therefore less than $\beta' - \beta$. Then the irrational number $\beta - m + \gamma$ lies between β and β' and therefore between α and α' . For from $m < \gamma < n$, we get $0 < \gamma - m < n - m < \beta' - \beta$. $\therefore \beta + \gamma - m < \beta'$ and $\gamma - m$ being > 0 , $\beta + \gamma - m > \beta$. Thus $\beta < \beta - m + \gamma < \beta'$. Having found one irrational number, we can find as many as we please.]

9. Define the number whose square is 3 by the method of sections.

10. Show that $\frac{p+3q}{p+q}$ is a closer approximation to $\sqrt{3}$ than $\frac{p}{q}$.

11. Show that if every rational number whose square is less than 2 is put in the lower class and every rational number whose square is greater than 2 is put in the upper class, this division is not a section.

12. State *without proof* Dedekind's theorem and indicate the difference between a section of rational numbers and a section of real numbers.

Show that if k is an approximation to $\sqrt{2}$, $\frac{4+3k}{3+2k}$ is a nearer one.

(C. H. 1965)

13. Explain how Dedekind defined a real number by a section of rational numbers and state when the number so defined is rational or irrational.

(C. H. 1965)

Show that $\sqrt{5}$ is not a rational number.

[If $\left(\frac{p}{q}\right)^2 = 5$, or $p^2 = 5q^2$, this may be written as $(5q-2p)^2 = 5(p-2q)^2$.

$\therefore \left(\frac{5q-2p}{p-2q}\right)^2 = 5$. Now since $1^2=1$, $2^2=4$, $3^2=9$, we have
 $2 < \frac{p}{q} < 3$, or, $2q < p < 3q$, so that $0 < p-2q < q$. Hence there is another fraction $\frac{5q-2p}{p-2q}$ which is equal to $\frac{p}{q}$ and whose denominator is less than q . This contradicts the assumption that $\frac{p}{q}$ is in its lowest terms. Thus, there is no rational number $\frac{p}{q}$ in its lowest terms whose square is 5.]

14. Define a real number as formulated by Dedekind and give an illustrative example.

(C. H. 1964)

15. Prove the existence of irrational real numbers by showing

that if m be a positive integer, which is not the square of another integer, then there is no rational number whose square is equal to m

(C. H. 1962)

[*Solution.* It is given that m is not the square of an integer.

We suppose then that m is the square of a rational number $\frac{p}{q}$ in its

lowest terms, i.e., $p^2 - mq^2 = 0$. Given a rational number, we can always find two consecutive integers between which it lies; let then

$\lambda < \frac{p}{q} < \lambda + 1$, or $\lambda q < p < (\lambda + 1)q$, whence $p - \lambda q < q$. We have the identity

$$(mq - \lambda p)^2 - m(p - \lambda q)^2 = (\lambda^2 - m)(p^2 - mq^2) = 0.$$

From this it follows that m is the square of the rational number $\frac{mq - \lambda p}{p - \lambda q}$, of which the denominator is less than q and this contradicts

the hypothesis that m is the square of the fraction $\frac{p}{q}$ which is in its

lowest terms. It is thus seen that there exists no rational number of which the square is m .

We are thus led to postulate the existence of a real number—not a rational number—the square of which is m .]

16. Show that no positive integer m other than a square number has a square root within the aggregate of rational numbers.

How is $\sqrt{m}$ defined in such a case? (C. H. 1964)

17. Explain the difference in the concepts of the ordinary rational number 2 and the *real* rational number 2, explaining why the latter is called rational. (C. H. 1963)

18. If a, b, x, y are rational numbers such that

$$(ay - bx)^2 + 4(a - x)(b - y) = 0,$$

prove that either $x = a$, $y = b$, or $1 - ab$ and $1 - xy$ are the squares of rationals. (C. H. 1962)

[*Hint:* Put $x - a = X$, $y - b = Y$; then $(aY - bX)^2 + 4XY = 0$.

If X, Y are not zero, solve for $\frac{X}{Y}$ and show that $1 - ab$ is a perfect square. Again the given equation is unaltered by the interchange

of x and a , y and b ; it follows that $1-xy$ is the square of a rational.]

19. If α , β , γ are real numbers such that $\alpha < \beta$ and $\beta < \gamma$, prove that $\alpha < \gamma$.

When are two *real* numbers said to be equal?

Define (a) the *real* number 0 and (b) the sum of two real numbers and show that if α be any real number, $\alpha+0=\alpha$.

(C. H. 1962)

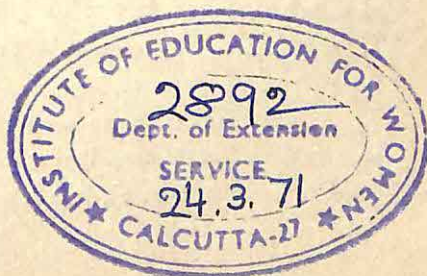
[To prove that $\alpha+0=\alpha$. Let $\alpha=(L_1, R_1)$, $0=(L_2, R_2)$ and $\alpha+0=(L, R)$. We have to show that L and L_1 are identical. A number of L is formed by adding some number a_1 of L_1 to a number of L_2 , which being negative, the sum is less than a_1 and therefore belongs to L_1 . Thus every member of L is a member of L_1 .

Let b_1 be a number of L_1 . As L_1 has no greatest number, there is a number b_1+k of L_1 greater than b_1 , k being a positive rational. If b_1+k of L_1 be added to $-k$ of L_2 , we find that b_1 is a member of L . Thus every member of L_1 is a member of L . So L and L_1 are identical.]

20. State when a real number α (as defined by a Dedekind section) is greater than the real number β and deduce that the real number $\sqrt{3}$ is greater than the real number $\sqrt{2}$. (C. H. 1961)

21. Show that there is no rational number whose m th power is equal to $\frac{a}{b}$ where $\frac{a}{b}$ is a positive fraction in its lowest terms unless a and b are perfect m th powers.

[Suppose that $\left(\frac{x}{y}\right)^m = \frac{a}{b}$, where $\frac{x}{y}$ is a positive fraction in its lowest terms. Then $bx^m = ay^m$. Since y is prime to x , y^m must be a factor of b . Let $b = ky^m$, where k is an integer. Then $a = kx^m$. Hence $k=1$, as a is prime to b . Therefore $a = x^m$, $b = y^m$, i.e., a and b are perfect m th powers.]



CHAPTER 3

AGGREGATES OR SETS

3.1. Aggregate or Set.

An aggregate or a set is defined as "a collection into a whole, of definite, well distinguished objects of our perception or of our thought".—*Cantor*.

In Analysis we usually consider aggregates of numbers possessing a common characteristic.

A system of real numbers or the points on a straight line corresponding to them, defined in any manner whatever, is called an aggregate or set of points in one dimension.

The objects or numbers which form a set are called its *elements*. Sets are sometimes represented by putting the elements within brackets. Thus a set formed of the numbers 1, 17, 3, 8 is denoted by $\{1, 17, 3, 8\}$.

If the number of elements of a set is finite, it is a *finite* set; if however, the number of elements is unlimited, it is an *infinite* set. For example, the set of integers from 1 to 100 is a finite set of 100 elements; the totality of odd numbers is an infinite set, as it contains infinitely many elements.

For a set, the order of succession of its elements shall not matter, if nothing is said to the contrary. Thus the set $\{1, 2, 3\}$ consisting of the elements 1, 2, 3, is the same set as $\{3, 1, 2\}$ or $\{2, 3, 1\}$. Further the elements of a set must be all distinct; thus the number complex 1, 2, 1, 3, 3 gives rise to one set only, namely the set $\{1, 2, 3\}$.

3.2. Real variable.

With regard to a set of real numbers, we can think of each element of the set individually or we can think of the set as a whole. If we think of the elements individually, we get different specified numbers or *constants*.

A *constant* is thus a specified number which has the same value in any set of mathematical operations.

On the other hand, when we think of the set as a whole and x

denotes any unspecified number of the set, and may represent each of the members of the set in turn, we call x a *real variable*. The individual numbers which x represents in succession are called its values. The set of values which a variable can assume constitutes what is called the *domain* of the variable.

If x assumes in succession every numerical value of a set of real numbers from a given number a to another given number b ($a < b$), then x is called a *continuous real variable* and x is said to vary in the domain or interval (a, b) . It is also represented by $a \leq x \leq b$. If x can assume both the end values a and b , the domain or interval (a, b) is said to be *closed*. If x cannot assume the value a , the interval is *open* at one end and this is indicated as $a < x \leq b$. If x cannot assume both the end values a and b , the interval is open at both ends, or is simply an *open interval* and is indicated as $a < x < b$.

3.3. Bounds of a Set.

Given a set S of real numbers, if there is a number m_1 greater than any number of the set, the set is said to be *bounded above* and m_1 is called an *upper bound* of the set. If $m_2 > m_1$, then m_2 is also another upper bound of the set.

With regard to this set of numbers, all the real numbers can be arranged in two classes, L and R , a number being put in the R -class, if it is an upper bound of the set, that is, if it is greater than any number of the set; it is put in the L -class if it is not an upper bound of the set, that is, if it is less than or equal to a number of the set. As the set is bounded above, both classes exist and are mutually exclusive. Further a member of R , being greater than a member of the set, is greater than a member of L which is either less than or equal to a member of the set. Hence by Dedekind's theorem (Art. 2'15), there is a number m such that all numbers less than it belong to L and all numbers greater than it belong to R . It is therefore the least of the upper bounds and is called *the upper bound* or the '*least upper bound*' (*l. u. b.*) of the set.

Again if there is a number l_1 less than any number of the set S , the set is said to be *bounded below* and l_1 is called a *lower bound* of the set. If $l_2 < l_1$, then l_2 is another lower bound.

A division of the real numbers in two classes, the lower class containing all numbers less than any number of the set, the upper

class containing all numbers greater than or equal to numbers of the set, would be a section, defining a number l , greater than all the numbers of the lower class, that is, greater than all the lower bounds. It is therefore the greatest among the lower bounds and is called the lower bound or the 'greatest lower bound' (*g. l. b*) of the set.

A set which has an upper bound and a lower bound is a *bounded* set.

If the bounds are members of the set, the bounds are said to be *attained*.

3.4. Properties of the bounds.

(a) The upper bound m of a set has the following properties :

(i) No number of the set is greater than m ;

(ii) there is a number of the set greater than $m - \epsilon$, however small the positive number ϵ may be.

To prove (i), it may be observed that if x a number of the set be greater than m , it would belong to the R -class and this is impossible. To prove (ii), it may be noted that if there be no number of the set greater than $m - \epsilon$, then $m (> m - \epsilon)$ would be greater than any number of the set and would thus belong to the R -class ; so m would not define the section.

(b) The lower bound l of a set has the following properties :

(i) No number of the set is less than l ;

(ii) there is a number of the set less than $l + \epsilon$, however small the positive number ϵ may be.

These are proved in exactly the same way as (a) above.

Note. The upper and lower bounds may belong to the set. This certainly occurs when the set is finite. But for an infinite set, the bounds need not belong to it. For example, consider the infinite set formed of the positive rational numbers whose squares are not greater than 2. The set is bounded above, its upper bound being the irrational number $\sqrt{2}$, which does not belong to the set. On the other hand, the set of positive real numbers, whose squares are not greater than 2, is bounded above and has $\sqrt{2}$ for its upper bound further $\sqrt{2}$ belongs to the set.

3.5. Sets of points. Definitions.

Let x, y, z be real variables. Every particular system of values of the variables is a *point* of which x, y, z are the coordinates. A

set of points has as many dimensions as a point of the set has coordinates. It is of one, two or three dimensions, that is, it is linear, superficial or spatial, according as a point of it has one, two or three coordinates.

A set of points is of one dimension or a linear set if a point of it has only one coordinate.

The distance between two points m and m' with coordinates (a, b, c) and (a', b', c') respectively is defined to be the quantity

$$\sqrt{(a-a')^2 + (b-b')^2 + (c-c')^2}.$$

The distance of a point m from a set S is the lower bound of the distance of the point m from a point of the set.

The distance between two sets, S_1 and S_2 , is the lower bound of the distance of a point of S_1 from a point of S_2 .

The ϵ -neighbourhood of a point m is the set of points at a distance $\leq \epsilon$. In the case of a linear set, it is the interval $(m-\epsilon, m+\epsilon)$.

m is an *isolated* point of S , if there are no other points of S in the ϵ -neighbourhood of m , ϵ being a positive number, however small.

m is a *limit* point of S , if every ϵ -neighbourhood of m contains infinitely many points of S . m may or may not belong to the set.

Every point of a set is either an isolated point or a limit point of the set.

The set S' of the limit points of a set S is called the *derived* set of S . The derived set may be the set itself.

3.6. Limit points of a set.

ξ is a limit point of a set S if every ϵ -neighbourhood of ξ contains at least one point of S other than ξ .

By making ϵ smaller and smaller, it is easy to see that if there be one point of S in a neighbourhood of ξ , there must be infinitely many points of S in it.

A limit point of a set is therefore a point about which an infinite number of points of the set cluster.

A limit point is also called a *limiting point*, a *cluster point*, a *point of accumulation* or a *point of condensation*.

Note. (i) A finite set cannot have a limit point.

(ii) An infinite set may or may not have a limit point. The infinite set of positive integers has no limit point.

(iii) Limit points of a set may or may not belong to the set.

(iv) A set may have more than one limit point. The rational numbers between 0 and 1 form a set with infinitely many limit points, since every point of the interval (0, 1), rational or irrational, is a limit point. Some of the limit points of the set belong to it and some, namely the irrational points of the interval and the end-points, do not.

In the set $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$, the lower bound 0 is a limit point but does not belong to the set; the upper bound 1 belongs to the set but is not a limit point.

The set of real numbers from 0 to 1 inclusive is a set which is identical with its derived set.

(v) If the set is bounded, it must have at least one limit point.

3.7. Bolzano-Weierstrass, theorem.

An infinite set, bounded above and below, has at least one limit point.

Let the infinite set S be bounded and have m and l for its upper and lower bounds, so that every member of S belongs to the interval (l, m) .

Relative to the set S , we can arrange all the real numbers in two classes, L and R , in the following manner. A number x will be put in L , if it exceeds no member or only a finite number of members of S ; it will be put in R , if it exceeds an infinite number of members of S .

Clearly each number has a class. Also l belongs to L and m to R as it is greater than an infinite number of members of S . So each class exists. A number of L is less than an infinite number of members of S whereas a number of R is greater than an infinite number of members of S ; thus any number of L is less than any number of R .

The two classes therefore determine a section of the real numbers, defining a number λ separating the two classes. Now ϵ being a positive number, however small, $\lambda - \epsilon$ which belongs to L can exceed only a finite number of members of S while $\lambda + \epsilon$ belonging to R must exceed an infinite number of members of S . Accordingly there must be an infinite number of members of S in the interval $(\lambda - \epsilon, \lambda + \epsilon)$. Hence λ must be a limit point of S .

Alternative Proof.

Let PQ represent the interval (l, m) . If we divide PQ into two equal parts, one at least of them must contain infinitely many points of S . We select the one which does, or, if both do, we select any one of them and call it P_1Q_1 .

If we similarly divide P_1Q_1 into two equal halves, one at least of them will contain infinitely many points of S . We select a half which does so and call it P_2Q_2 .

Proceeding in this way, we get a set of intervals

$$PQ, P_1Q_1, P_2Q_2, \dots, P_nQ_n, \dots$$

each of which is half of its predecessor and each of which contains infinitely many points of S .

The points $P, P_1, P_2, \dots, P_n, \dots$ progress steadily from left to right and P_n can never go beyond Q . Again $Q, Q_1, Q_2, \dots, Q_n, \dots$ progress steadily from right to left and Q_n cannot go beyond P .

Now $P_nQ_n = \frac{1}{2^n}PQ$, so that $P_nQ_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore P_n and Q_n ultimately approach the same point T . Then T is a limit point of S . For suppose ξ to be the coordinate of T and consider any interval $(\xi - \epsilon, \xi + \epsilon)$. If n is sufficiently large, P_nQ_n will lie entirely within this interval; this will certainly be the case as soon as $\frac{1}{2^n}PQ < \epsilon$. Hence $(\xi - \epsilon, \xi + \epsilon)$ contains infinitely many points of S ; in other words, ξ is a limit point of S .

3.8. Fundamental operations with sets.

Equal Sets. Two sets S and R are said to be equal (in symbols $S=R$), if each element of S is an element of R and conversely. $S \neq R$ means that S and R are not equal sets. The symbol $x \in S$ means that x is an element of S . For equal sets, if $x \in S$ then $x \in R$ and if $x \in R$, then $x \in S$.

Subsets. R is a *proper* subset of S , if every element of R is an element of S and $R \neq S$, that is, S has one or more elements not contained in R . In symbols this is denoted by $R \subset S$ or $S \supset R$.

Each of two equal sets is sometimes called an *improper* subset of the other.

Null Sets. It is sometimes convenient to speak of a set with no elements as a *null* set, a *void* set or an *empty* set. Such a set is denoted by 0 or ϕ .

Union or Join of Sets.

The union or join of two sets R_1 and R_2 is a set S consisting entirely of the elements of R_1 and R_2 , the common elements, if any, being taken only once. Symbolically, $S = R_1 \cup R_2$. If R_1 has m elements and R_2 has n elements, the number of elements of $S \leq m+n$ according as R_1 and R_2 have common elements or are mutually exclusive.

This definition of the union of sets can be extended to more than two sets. The union of a number of null sets is a null set.

Intersection or Meet of Sets.

The intersection or the meet of two sets R_1 and R_2 is the set S formed of the common elements of R_1 and R_2 . In symbols, $S = R_1 \cap R_2$.

If R_1 and R_2 have no common elements, the intersection of R_1 and R_2 is a null set. Sets with no common elements are called *mutually exclusive* or *disjoint* or *disjunct*. If one or both of R_1 and R_2 are null sets, their intersection is a null set.

The definition of the intersection of sets can be extended to more than two sets.

Note. The union of two sets R_1 and R_2 is sometimes called their *sum* and represented in symbols by $R_1 + R_2$. The intersection of two sets R_1 and R_2 is sometimes called their *product* and represented by $R_1 \cdot R_2$ or $R_1 R_2$. (Valle'o Poussain = *Inte'grales de Lebesgue*. Rogosinski—*Volume and Integral*) In what follows, we shall adopt these simpler notations.

In their union and intersection, sets obey the following laws of algebra :

1. Commutative law : $S_1 + S_2 = S_2 + S_1$; $S_1 S_2 = S_2 S_1$.
2. Associative law : $(S_1 + S_2) + S_3 = S_1 + (S_2 + S_3)$
 $= S_1 + S_2 + S_3$.
 $(S_1 S_2) S_3 = S_1 (S_2 S_3) = S_1 S_2 S_3$.
3. Distributive law : If $S = S_1 + S_2$ and R any other set, then $SR = S_1 R + S_2 R$.

For if $x \in SR$, then $x \in S$ and also $x \in R$. Now if $x \in S$, then x belongs to S_1 or S_2 or to both. Hence if $x \in S_1$, then $x \in S_1 R$; again if $x \in S_2$, then $x \in S_2 R$. Thus if $x \in SR$, it belongs to $S_1 R$ or to $S_2 R$ or to both ; that is, it belongs to $S_1 R + S_2 R$.

Conversely, if $x \in S_1R$, then $x \in S_1$ and also $x \in R$. So $x \in SR$, whether it belongs to S_2R or not. Again if $x \in S_2R$, then $x \in S_2$ and also $x \in R$, i.e., $x \in SR$. Thus $SR = S_1R + S_2R$.

3'9. Complement of a Set.

Let $R \subset S$. The complement CR of R in S is the set of elements of S which are not in R . It is also denoted by $S - R$. Since each of R and CR may be looked upon as the complement of the other, we have $C(CR) = R$.

If $S_1 + S_2 = S$ and $S_1S_2 = 0$, then $S_1 = CS_2$ and $S_2 = CS_1$.

We have the following theorems :

Theorem I. Let R_1 and R_2 be two subsets of S . Then

$$C(R_1R_2) = CR_1 + CR_2.$$

Let $x \in C(R_1R_2)$. Then x does not belong to R_1R_2 ; so x does not belong to at least one of R_1 and R_2 . If R_1 does not contain x , then $x \in CR_1$. If R_2 does not contain x , $x \in CR_2$. Thus x belongs to either CR_1 or to CR_2 or to both. Therefore $x \in CR_1 + CR_2$. Hence

$$C(R_1R_2) \subset CR_1 + CR_2. \quad \dots \quad (1)$$

Again let $x \in CR_1 + CR_2$; then x belongs to CR_1 or to CR_2 or to both. Therefore x does not belong to either R_1 or to R_2 or to any of R_1 and R_2 . Hence x cannot belong to R_1R_2 . So $x \in C(R_1R_2)$. Therefore $CR_1 + CR_2 \subset C(R_1R_2)$. $\dots \dots \dots$ (2)

From (1) and (2) it follows that $C(R_1R_2)$ and $CR_1 + CR_2$ are identical. Thus

$$C(R_1R_2) = CR_1 + CR_2.$$

$$\text{or, } C(R_1 \cap R_2) = (CR_1) \cup (CR_2).$$

Theorem II. Let R_1 and R_2 be two subsets of S . Then

$$C(R_1 + R_2) = CR_1 \cdot CR_2.$$

If x is in S but not in $R_1 + R_2$, then it is neither in R_1 , nor in R_2 ; so it is in both CR_1 and CR_2 and therefore in $CR_1 \cdot CR_2$. Hence every element of $C(R_1 + R_2)$ is an element of $CR_1 \cdot CR_2$ and so $C(R_1 + R_2) \subset CR_1 \cdot CR_2$. $\dots \dots \dots$ (1)

Again if y be an element of S which is in $CR_1 \cdot CR_2$, then it is neither in R_1 , nor in R_2 and therefore not in $R_1 + R_2$; so it is in $C(R_1 + R_2)$. Hence every element of $CR_1 \cdot CR_2$ is an element of $C(R_1 + R_2)$ and so $CR_1 \cdot CR_2 \subset C(R_1 + R_2)$. $\dots \dots \dots$ (2)

From (1) and (2) it follows that $C(R_1 + R_2)$ and $CR_1.CR_2$ are identical. Thus

$$C(R_1 + R_2) = CR_1.CR_2,$$

or, $C(R_1 \cup R_2) = (CR_1) \cap (CR_2).$

Note. Each of the above theorems can be deduced from the other by taking complements.

3'10. Open and closed sets in an interval. Definitions.

Let S be a set of points in an interval Δ , CS its complement.

A point m is an *interior* point of S in Δ , if it is at a non-zero distance from CS , that is, if for some ϵ , an ϵ -neighbourhood of it is entirely contained in S .

The point m is an *exterior* point of S in Δ , if it is an interior point of CS , that is, there is no neighbourhood of m containing points of S .

m is a *frontier* point of S in Δ , if it is neither an interior point nor an exterior point of S ; in other words, if it is at a zero distance from both S and CS .

Every point of a set is either an isolated point or a limit point of the set. Every interior point is of course a limit point of the set.

A set S in Δ is *open* if every point of the set is an interior point, that is, if each point m of the set has, for some ϵ , an ϵ -neighbourhood entirely in the given set.

A set S in Δ is *closed* if every limit point of the set belongs to it, or if the derived set S' of S is a subset of it.

Remarks :

1. A closed interval is a closed set, whereas an open interval is an open set.
2. The set of rational numbers is not closed.
3. Every finite set is a closed set.
4. The concepts of closed and open sets are not mutually exclusive; there may be a set which is both open and closed and a set which is neither open nor closed. The set of real numbers is an open as well as a closed set. The set of rational numbers is neither open nor closed. A null set may be looked upon as a closed set and also an open set.

A set which is identical with its derived set is called a *perfect* set.

3.11. Properties of Open and Closed Sets.

(a) If a set R be open in S , its complement CR is closed in S and conversely.

Let $R \subset S$. We first prove that the complement CR of R in S is closed if, and only if, R is open in S .

First suppose CR is closed in S ; then CR contains all its limit points and so no point of R is such that every neighbourhood of it contains a point of CR . Hence every point of R has a neighbourhood which contains no point of CR and which is, therefore, entirely in R . Consequently R is open in S .

Next suppose that R is open in S . Therefore every point of R has a neighbourhood which contains no point of CR and so no limit points of CR . Hence all the limit points of CR are contained in CR . So CR is closed in S .

Thus CR is closed in S if, and only if, R is open in S .

The converse proposition that R is closed in S if, and only if, CR is open in S can be proved similarly or by taking complements.

(b) The union and intersection of a finite number of open sets are each an open set.

It will be sufficient to consider two open sets O_1 and O_2 .

(i) Let $S = O_1 + O_2$. If $x \in S$, then it belongs to O_1 or to O_2 or to both. Suppose $x \in O_1$. O_1 being open, there is a neighbourhood of x entirely contained in O_1 and therefore in S . Thus every point of S has a neighbourhood entirely contained in S . So S is open.

(ii) Next let $P = O_1 O_2$. If $x \in P$, then it belongs to both O_1 and O_2 . As $x \in O_1$, there is a neighbourhood N_1 of x wholly contained in O_1 as O_1 is open. Again as $x \in O_2$ and O_2 is open, there is a neighbourhood N_2 of x wholly contained in O_2 . The intersection N of the neighbourhoods N_1 and N_2 is a neighbourhood of x wholly contained in both O_1 and O_2 and therefore in $O_1 O_2$. Thus every point of P has a neighbourhood wholly contained in P . Thus $P = O_1 O_2$ is open.

(c) The union and intersection of a finite number of closed sets are each a closed set.

It will be sufficient to consider two closed sets U_1 and U_2 .

Since U_1 and U_2 are closed, CU_1 and CU_2 are open, by (a) above.

Again by (b) above, CU_1 , CU_2 and $CU_1 + CU_2$ are open sets. So the complements of these are closed sets.

$$\text{Now } C(CU_1, CU_2) = C(CU_1) + C(CU_2) = U_1 + U_2.$$

[Theorem I, Art. 3'9]

$$\text{And } C(CU_1 + CU_2) = C(CU_1) \cdot C(CU_2) = U_1 U_2.$$

[Theorem II, Art. 5'9]

Thus $U_1 + U_2$ and $U_1 U_2$ are closed sets.

Or, we may proceed thus :

(i) Let $S = U_1 + U_2$. Let x be a limit point of S . Then every neighbourhood of x contains an infinity of points of S and so an infinity of points of at least one of U_1 and U_2 , say U_1 . For otherwise there would be a neighbourhood of x which would not contain more than a finite number of points of S , which would mean that x is not a limit point of S , contrary to our hypothesis. Hence there is at least one set U_1 such that every neighbourhood of x contains an infinity of points of U_1 , so that x is a limit point of U_1 . As U_1 is closed, x belongs to U_1 and therefore to S . Thus S is closed.

(ii) Let $P = U_1 U_2$. If x be a limit point of P , every neighbourhood of x must contain an infinity of points of P and so an infinity of points of U_1 and U_2 . Therefore x is a limit point of each of U_1 and U_2 and as these are closed sets, x must belong to both U_1 and U_2 and therefore to $U_1 U_2$. Thus $P = U_1 U_2$ is a closed set.

(d) The derived set of any set is a closed set.

Let S be a given set and S' its derived set. Let ξ be a limit point of S' ; then there are points η of S' at a distance less than $\frac{1}{2}\epsilon$ from ξ . Again η , being a point of S' , is a limit point of S ; so there are points ζ of S at a distance less than $\frac{1}{2}\epsilon$ from η . Thus there are points ζ of S at a distance less than ϵ from ξ ; so ξ is a limit point of S and therefore belongs to S' . Thus every limit point of S' belongs to S' . In other words, the derived set S' is closed.

3'12. Enumerable and Non-enumerable Sets.

An infinite set is said to be *enumerable* or *countable* if a one-to-one correspondence can be set up between its elements and the positive integers 1, 2, 3, ..., n , ...; in other words, an infinite set is enumerable, if its elements can be arranged in the form of a

sequence*: $u_1, u_2, u_3, \dots, u_n, \dots$, so that every element of the set occupies a definite place in the sequence.

An infinite set which is not enumerable is *non-enumerable* or *uncountable*.

A *point of condensation* of a set S is a point ξ in an arbitrary neighbourhood of which there is a non-enumerable infinity of points of the set. (*Lindelöf*).

3'13. Upper and Lower limits of a Set.

When a set S has more than one limit point, the greatest limit point μ is called the *upper limit* of S and the least limit point λ is called the *lower limit* of S . If ϵ be a positive number, however small,

- (i) an infinite number of members of S are greater than $\mu - \epsilon$;
- (ii) a finite number, at the most, of members of S are greater than $\mu + \epsilon$;
- (iii) an infinite number of members of S are smaller than $\lambda + \epsilon$;
- (iv) a finite number, at the most, of members of S are smaller than $\lambda - \epsilon$.

3'14. Oscillation of a bounded Set.

The difference $m - l$ between the upper bound m and the lower bound l of a bounded set is called the *oscillation* of the set.

3'15. Measure of a Set.

Let S be a set of points in the interval δ . Let the points of S be enclosed in an enumerable set of sub-intervals according to some fixed rule. The sub-intervals should not overlap, but some may abut and some even may reduce to points. These sub-intervals form a convergent series as their sum cannot exceed the length of δ . The lower bound of such sums is called the *measure* of the set. The measure of a set is never negative.

If the set be enumerable, its measure is zero.

3'16. Some theorems on enumerable and non-enumerable sets.

We proceed to prove certain theorems about enumerable and non-enumerable sets.

* A 'sequence' is an ordered set. See next chapter.

(a) The sum of two enumerable sets is enumerable.

Let $S : \{u_1, u_2, u_3, \dots\}$ and $T : \{v_1, v_2, v_3, \dots\}$ be two enumerable sets, which we suppose to be mutually exclusive. Then $S+T$ which contains the elements of both sets may be written as $S+T : \{w_1, w_2, w_3, \dots\}$ where $w_{2n-1} = u_n$ and $w_{2n} = v_n$. Thus $S+T$ can be written in the form of a sequence and hence $S+T$ is enumerable.

The theorem can be extended to any finite number of sets,

(b) A proper infinite subset of an enumerable set is enumerable.

Let $S : \{u_1, u_2, u_3, \dots\}$ be a given enumerable set and suppose R is an infinite subset of S . Then among the elements of S , there is a first element u_{k_1} which belongs to R ; let $v_1 = u_{k_1}$. This is followed by a next element u_{k_2} of S , which is also an element of R ; let $v_2 = u_{k_2}$. This process is continued and never terminates. We have however arranged the elements of R as a sequence: $v_1, v_2, v_3, \dots$, so that R is enumerable.

(c) The set of all rational numbers is enumerable.

We first consider the positive rational numbers. In the first line we write all whole numbers, i.e., all numbers with denominator 1; in the next line, we write all fractions with denominator 2; in the third line, fractions with denominator 3; and so on. We thus get the following scheme of numbers;

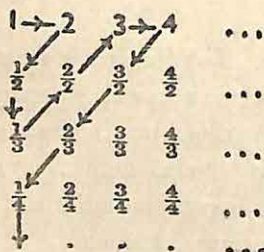


Fig. 4

If we write down the numbers in the order of succession indicated by the line drawn (leaving out numbers which have already appeared), then we get every positive rational number once and only once. The totality of these rational numbers is thus written as the sequence

$$1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \dots$$

If this sequence be denoted by $\{u_1, u_2, u_3, \dots\}$, then obviously the sequence $\{0, -u_1, u_1, -u_2, \dots\}$ is the set of all rational numbers which is therefore enumerable.

(d) The set of real numbers in the interval $0 \leq x \leq 1$ is non-enumerable.

We assume in the first place that the set of real numbers

$0 \leq x \leq 1$ is enumerable. A contradiction will show that our assumption is wrong.

Since by supposition the set $0 \leq x \leq 1$ is enumerable, by theorem (b) above, the subset $0 < x \leq 1$ is enumerable. It consists of the positive proper fractions and the number 1. Now $1 = .999\cdots$ can be expressed uniquely as a non-terminating decimal. Again the terminating decimals among the positive proper fractions can be expressed uniquely as non-terminating decimals by reducing the last digit on the right by unity and adjoining an endless chain of 9's. (e.g., $.05 = .04999\cdots$, $.72 = .71999\cdots$). Thus all numbers in the set $0 < x \leq 1$ can be expressed as non-terminating decimals. And as by supposition they form an enumerable set, they can be arranged as a sequence as in the scheme below :

$$\begin{array}{ccccccccc}
 u_1 & = & 0 & a_{11} & a_{12} & a_{13} & a_{14} & \dots & \\
 u_2 & = & 0 & a_{21} & a_{22} & a_{23} & a_{24} & \dots & \\
 u_3 & = & 0 & a_{31} & a_{32} & a_{33} & a_{34} & \dots & \\
 \dots & & \dots & \dots & \dots & \dots & \dots & \dots & \dots
 \end{array}$$

Fig. 5

From the principal diagonal indicated by the line, we form the non-terminating decimal $0.a_{11}a_{22}a_{33}\dots$, and from this we construct a new non-terminating decimal by replacing each digit a_{nn} by a different digit b_n , different from 0, giving the decimal fraction $u = 0.b_1b_2b_3\dots b_n\dots$. u does not terminate as none of the b 's is 0; also $0 < u \leq 1$. If $0 < x \leq 1$ be enumerable, u will have to coincide in all its digits with one of the decimal fractions in the scheme above. This however is not possible according to our construction of u , for u certainly differs from each u_n in the n th digit. Thus all the numbers in $0 < x \leq 1$ cannot be arranged as a sequence. So the set $0 < x \leq 1$ is not enumerable. Hence the real numbers in $0 \leq x \leq 1$ form a non-enumerable set.

Examples :

1. If x be any member of a bounded aggregate S_1 whose upper and lower bounds are B_1 and b_1 , find the upper and lower bounds of the aggregate S_2 whose members are $-x$. (C. H. 1965)

$$[\text{As } x \geq b_1, \text{ we have } -x \leq -b_1, \quad \dots \quad (1)]$$

where $-x$ is any member of S_2 .

If ϵ be any positive number however small, there exists a number x of S_1 such that $x < b_1 + \epsilon$; this shows that there exists a number $-x$ of S_2 such that $-x > -b_1 - \epsilon$ (2)

Hence from (1) and (2), $-b_1$ is the upper bound of S_2 .

It may be shown similarly that $-B_1$ is the lower bound of S_2 .]

2. x and y are any two members of a bounded aggregate S ; find the upper bound of the aggregate S_1 of members $x - y$. (C. H. 1965)

[Let B, b be the upper and lower bounds of the aggregate S .

We have $x \leq B$ and $y \geq b$, i.e., $-y \leq -b$. $\therefore x - y \leq B - b$... (1)

Again $x > B_1 - \frac{\epsilon}{2}$ and $y < b + \frac{\epsilon}{2}$, i.e., $-y > -b - \frac{\epsilon}{2}$.

$\therefore x - y > B - b - \epsilon$ (2)

From (1) and (2) we see that $B - b$ is the upper bound of members $x - y$ of the aggregate S_1 .]

3. x and y are any two members of bounded aggregates S and S' ; find the upper bound of the aggregate S_1 of numbers $x - y$.

[Let B_1, b_1 be the upper and lower bounds of S and B_2, b_2 those of S' .

We have $x \leq B_1$ and $y \geq b_2$, i.e., $-y \leq -b_2$. $\therefore x - y \leq B_1 - b_2$... (1)

Again $x > B_2 - \frac{\epsilon}{2}$ and $y < b_2 + \frac{\epsilon}{2}$, i.e., $-y > -b_2 - \frac{\epsilon}{2}$.

$\therefore x - y > B_1 - b_2 - \epsilon$ (2)

Hence from (1) and (2), $B_1 - b_2$ is the upper bound of the aggregate S_1 of numbers $x - y$.

It may be shown similarly that $b_1 - B_2$ is the lower bound of S_1 .]

4. x is any member of a bounded set whose upper and lower bounds are B_1 and b_1 , and y any member of a bounded set with upper and lower bounds B_2 and b_2 . Show that the bounds of the set of numbers $x + y$ are $B_1 + B_2$ and $b_1 + b_2$.

5. B_1 and b_1 are the bounds of an aggregate S_1 of positive numbers x , and B_2, b_2 are those of another aggregate S_2 of positive numbers y . Show that $B_1 B_2$ and $b_1 b_2$ are the bounds of the aggregate S of numbers xy .

6. Prove that every aggregate bounded above has an upper bound. (C. H. 1965)

7. Define the limiting point of an aggregate. If S is an infinite aggregate,

(i) can S have no limiting point?

(ii) can every member of S be a limiting point?

Illustrate your answers by examples. (C. H. 1965)

[(i) The set of integers though infinite has no limiting point.

(ii) Every point of the set of real numbers $0 \leq x \leq 1$ is a limiting point.]

8. State and prove Bolzano-Weierstrass' theorem on limiting points of an aggregate on the real axis. (C. H. 1965)

9. Give an example of a set

(i) which is identical with its limiting points;

(ii) whose limiting points do not belong to the set.

[(ii) The set $1+\frac{1}{2}, -1-\frac{1}{2}, 1+\frac{1}{3}, -1-\frac{1}{3}, 1+\frac{1}{4}, -1-\frac{1}{4}, \dots$ has two limiting points, 1 and -1 , not belonging to the set.]

10. Find the limiting points of

(i) the aggregate of numbers $\frac{1}{m} + \frac{1}{n}$, where m and n take all integral values; (C. H. 1966)

(ii) the aggregate of numbers $\frac{1}{m} + \frac{1}{n} + \frac{1}{p}$, where m, n, p take all integral values.

[(i) $0, \frac{1}{m}$; (ii) $0, \frac{1}{m}, \frac{1}{m} + \frac{1}{n}$]

11. Define (i) the lower bound and (ii) the lower limit of an aggregate and show by illustrative examples that these may be either equal or unequal. (C. H. 1964)

12. Define the exact upper bound of a bounded infinite aggregate, and show by examples that this may or may not be equal to the upper limit. (C. H. 1961)

13. Exhibit the aggregate of all rational numbers between 0 and 1 as a sequence. (C. H. 1964, 1965)

[First consider the rational numbers in the open interval $0 < x < 1$. Let q be any integer > 1 . In $\frac{p}{q}$, give to p the values $1, 2, 3, \dots, q-1$. From the $q-1$ fractions thus obtained, we suppress those that are reducible, i.e., those for which p and q are not prime to each other; they are obviously finite in number. If we do this

in succession for $q = 2, 3, 4, \dots$, we get every time a finite number of rational numbers. We thus get the series

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots$$

giving all rational numbers between 0 and 1, without omission or repetition; these have been exhibited above as a sequence.

For rational numbers in $0 \leq x \leq 1$, we adjoin 0 and 1 to the above series and get the sequence

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots$$

14. The sum of an enumerably infinite number of enumerable sets is enumerable.

[As the number of sets is enumerable, they can be arranged as a sequence, *viz.*, $S_1, S_2, S_3, \dots$. Each of the sets being enumerable, each can be written as a sequence. We may thus write

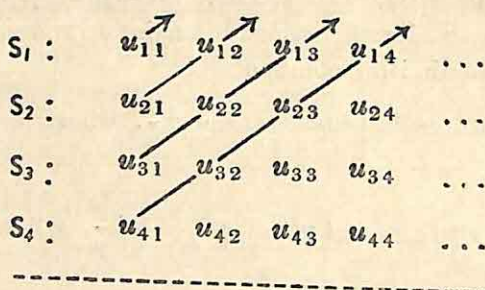


Fig. 6

We now write the elements anew in the order of succession indicated by the arrows, each arrow covering elements for which the sum of the suffixes is the same and each arrow covering a finite number of elements. We thus get

$$u_{11}, u_{21}, u_{12}, u_{31}, u_{22}, u_{13}, \dots$$

a sequence containing all the elements that occur in any of the given sets, thereby proving the theorem.

We may suppose that the sets have no common elements; if however, there be common elements, we may leave out elements that have already appeared.]

CHAPTER 4

SEQUENCE

4.1. Sequence.

If the members of an infinite aggregate of numbers can be arranged in the order of the natural numbers 1, 2, 3, ..., the aggregate so arranged is called a *sequence* of numbers.

A sequence is thus an ordered aggregate, the member corresponding to n is called the n th term of the sequence and is often represented by a_n or u_n , the sequence itself being then represented by $\{a_n\}$ or $\{u_n\}$. A sequence is said to be given when we know some law or laws by which any particular term of the sequence can be determined.

Examples :

- (i) $a_n = a$ for all n gives the sequence : $a, a, a, \dots$,
- (ii) $a_n = \frac{1}{n}$ gives the sequence : $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$,
- (iii) $a_n = (-1)^n$ gives the sequence : $-1, 1, -1, 1, \dots$,
- (iv) $a_n = \frac{(-1)^n}{n}$ gives $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots$,
- (v) $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$ gives $-2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, \dots$.

4.2. Limit of a Sequence : Convergent Sequence.

Consider the sequence $\{a_n\}$ where $a_n = \frac{2n+3}{n}$. If we choose any small positive number, '01 say, the difference $|a_n - 2|$ $= \left| \frac{2n+3}{n} - 2 \right| = \frac{3}{n}$ will be less than '01, if $n > 300$; i.e., $|a_n - 2| < '01$ for values of n from 301 onwards. Again for a still smaller number, '0001 say, $|a_n - 2| < '0001$, if $n > 30000$. However small the positive number we choose, we can always find a value of n such that $|a_n - 2|$ will be less than this chosen value for all values of n greater than this determinate value. As the arbitrary positive number may be chosen as small as we please, it is clear that the

difference $|a_n - 2|$ can be made as small we please for sufficiently large values of n . When that is so, the number to which a_n can be made to approach, is called the *limit* of the sequence $\{a_n\}$ as n tends to infinity.

As a rough definition of the limit of a sequence, we may say that a sequence tends to a limit A as n tends to infinity, if we can make the terms of the sequence approach A by making n sufficiently large.

It would be clear from the above example that the limit of a sequence may be defined as follows :

A sequence $\{a_n\}$ is said to tend to the limit A as n tends to infinity, if corresponding to any arbitrarily chosen positive number ϵ , however small, we can find a positive integer ν such that $|a_n - A| < \epsilon$ when $n > \nu$.

The notation employed to denote this is

$$\text{Lt } a_n = A, \text{ or } \lim_{n \rightarrow \infty} a_n = A,$$

or simply $\text{Lt } a_n = A, \lim a_n = A$.

When a sequence tends to a finite limit A , it is said to be *convergent* or to converge to the limit A .

A convergent sequence which tends to the limit 0 is called a *null sequence*.

4-3. Theorem : A convergent sequence cannot tend to two distinct limits.

If possible, let A and B be two distinct limits to which $\{a_n\}$ converges. Then corresponding to the arbitrary positive number $\frac{1}{2}\epsilon$, we can find integers ν_1 and ν_2 such that

$$|a_n - A| < \frac{1}{2}\epsilon \text{ for } n > \nu_1; \quad \dots \quad (1)$$

$$|a_n - B| < \frac{1}{2}\epsilon \text{ for } n > \nu_2. \quad \dots \quad (2)$$

If ν be the greater of ν_1 and ν_2 , both (1) and (2) hold for $n > \nu$.

$$\begin{aligned} \text{Now } |A - B| &= |(A - a_n) + (a_n - B)| \\ &\leq |a_n - A| + |a_n - B| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \\ &< \epsilon, \end{aligned}$$

for $n > \nu$; this is absurd as the difference between the two distinct fixed numbers cannot be made less than an arbitrary small number.

4.4. Divergent and Oscillatory Sequences.

(1) If corresponding to any arbitrary positive number M , however large, a positive integer ν can be found such that $a_n > M$ for $n > \nu$, we say that the sequence is *positively divergent* and that it diverges to $+\infty$ and we write

$$\lim_{n \rightarrow \infty} a_n = +\infty.$$

(2) If corresponding to a negative number $-M$, however large M may be, we can find a positive integer ν such that $a_n < -M$ when $n > \nu$, the sequence is *negatively divergent* and diverges to $-\infty$ and we write

$$\lim_{n \rightarrow \infty} a_n = -\infty.$$

(3) When the sequence does not converge and does not diverge to $+\infty$ or to $-\infty$, it is said to *oscillate*.

An oscillatory sequence is said to *oscillate finitely*, if there is a positive number K such that $|a_n| < K$ for all values of n ; and it is said to *oscillate infinitely* when there is no such number.

Examples :

The sequence for which $a_n = (-1)^n$ oscillates finitely, and the sequence for which $a_n = (-1)^n n$ oscillates infinitely.

Note. The last example shows that the terms of a sequence may ultimately be very large in absolute value, yet the sequence may not diverge to $+\infty$ or to $-\infty$.

4.5. Bounds of a Sequence.

As a sequence is obtained from an aggregate by ordering it, the sequence may be said to correspond to the aggregate from which it is obtained. If an aggregate is bounded and have M and m for its bounds, the corresponding sequence is bounded and have the same bounds. Further the bounds of the sequence have the same properties as the bounds of the aggregate.

The limiting points of the aggregate are limiting points of the corresponding sequence. The converse is not necessarily true; a sequence may have limiting points which are not limiting points of the aggregate from which it is derived.

The Bolzano-Weierstrass' theorem is also true for a bounded sequence and can be proved similarly.

The upper and lower limits of a sequence are defined similarly and have the same properties. The symbols for the upper and lower limits are :

$$\overline{\lim} a_n, \underline{\lim} a_n.$$

4.6. Theorem : Every convergent sequence is bounded.

If the sequence $\{a_n\}$ is convergent and converges to the limit a , we have

$$|a_n - a| < \epsilon \text{ for } n > \nu.$$

$$\text{or, } a - \epsilon < a_n < a + \epsilon, \text{ for } n > \nu.$$

Thus $a_{\nu+1}, a_{\nu+2}, \dots$ all lie between $a - \epsilon$ and $a + \epsilon$. Let g and G be the least and the greatest of the ν numbers $a_1, a_2, \dots, a_\nu$. Then if $m = \min. (a - \epsilon, g)$ and $M = \max. (a + \epsilon, G)$, we have

$$m \leq a_n \leq M,$$

for all values of n ; in other words $\{a_n\}$ is bounded.

4.7. Monotonic Sequence.

If the terms of a sequence satisfy either of the following two sets of relations :

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots,$$

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots,$$

the sequence is said to be monotonic. In the first case the terms never decrease and the sequence is *monotonic increasing* or *monotone ascending*; in the second case the terms never increase and the sequence is *monotonic decreasing* or *monotone descending*.

Note. If $a_1 < a_2 < \dots < a_n < \dots$, $\{a_n\}$ is monotone increasing in the *strict sense*; and if $a_1 > a_2 > \dots > a_n > \dots$, $\{a_n\}$ is monotone decreasing in the *strict sense*.

4.8. Theorem : A monotone increasing sequence either converges to a finite limit or diverges to $+\infty$ according as it is bounded above or not.

Suppose in the first place that the monotone increasing sequence $\{a_n\}$ is bounded above and K is an upper bound. The sequence is bounded by a_1 on the left and by K on the right. It must therefore have a least upper bound k , equal to or less than K , and however small the positive number ϵ may be, there will be a term of the

sequence a_p , greater than $k - \epsilon$. If we consider the aggregate formed by the distinct terms of the sequence, all its terms after a_{p-1} are to the right of $k - \epsilon$ and not to the right of k . If any of them coincide with k , from that stage on, the terms must be equal. We have therefore

$$|k - a_n| < \epsilon \text{ when } n \geq p;$$

so the sequence is convergent and has k for its limit.

If $\{a_n\}$ is not bounded above, however large the positive number M may be, $a_n > M$ for $n \geq p$. So $\{a_n\}$ diverges to $+\infty$.

Note. A monotone increasing sequence cannot oscillate.

4.9. Theorem : A monotone decreasing sequence either converges to a finite limit or diverges to $-\infty$ according as it is bounded below or not.

The proof is exactly similar to the above.

Note. A monotone decreasing sequence cannot oscillate.

4.10. General Principle of Convergence (Cauchy).

The test in Art. 4.2 verifies whether a given number A is the limit of the sequence $\{a_n\}$. In the absence of a knowledge of the limit of the sequence, it fails to determine whether the sequence is convergent or not. The test to determine the convergence of a sequence without a knowledge of its limit is furnished by the following test known as *Cauchy's General Principle of Convergence*.

The necessary and sufficient condition for the convergence of a sequence $\{a_n\}$ is : Corresponding to an arbitrarily chosen positive number ϵ , however small, it must be possible to find a positive integer m , so that for every positive integer p ,

$$|a_{n+p} - a_n| < \epsilon$$

when $n \geq m$ and $p > 0$.

(i) The condition is *necessary*. This means that if there is a limit l for the sequence, the above condition must be satisfied.

Since $a_n \rightarrow l$, corresponding to any positive number ϵ , we can find m so that

$$|a_n - l| < \frac{1}{2}\epsilon$$

when $n \geq m$. From this it follows at once that

$$|a_{n+p} - l| < \frac{1}{2}\epsilon$$

when $n \geq m$ and $p > 0$.

$$\begin{aligned}
 \therefore |a_{n+p} - a_n| &= |a_{n+p} - l + l - a_n| \\
 &\leq |a_{n+p} - l| + |a_n - l| \\
 &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \\
 &< \epsilon
 \end{aligned}$$

for $n \geq m$ and p any positive integer.

(ii) This condition is *sufficient*. This means that when the condition is satisfied, $\{a_n\}$ must converge to a limit l .

We shall first show that under the given condition, the sequence is bounded.

Let us give any particular value to ϵ , say η . There exists a positive integer ν such that

$$|a_{n+p} - a_n| < \eta \text{ when } n \geq \nu \text{ and } p > 0.$$

Taking $n = \nu$, we see that

$$|a_p - a_\nu| < \eta \text{ when } p > 0.$$

$$\text{i.e., } a_p - \eta < a_{\nu+p} < a_p + \eta \text{ when } p > 0.$$

This means that all the members of the sequence $\{a_n\}$ except perhaps the finite set of members $a_1, a_2, a_3, \dots, a_{\nu-1}$ lie between two fixed numbers $a_p - \eta$ and $a_p + \eta$. If k be the least and K the greatest of the finite set of numbers

$$a_1, a_2, a_3, \dots, a_{\nu-1}, a_p - \eta, a_p + \eta,$$

we see that

$$k \leq a_n \leq K,$$

for every value of n and so $\{a_n\}$ is bounded.

The sequence $\{a_n\}$ has therefore at least one limiting point l .

If possible, let there be also another limiting point l' .

Corresponding to any positive number ϵ , however small, there exists a positive integer m such that

$$|a_{n+p} - a_n| < \frac{1}{3}\epsilon \text{ when } n \geq m \text{ and } p > 0. \quad \dots \quad (1)$$

Also since l and l' are limiting points, there exist positive integers, s and t both $\geq m$ such that

$$|a_s - l| < \frac{1}{3}\epsilon, \quad |a_t - l'| < \frac{1}{3}\epsilon, \quad \dots \quad \dots \quad (2)$$

$$\text{Also from (1), } |a_s - a_t| < \frac{1}{3}\epsilon, \quad \dots \quad \dots \quad (3)$$

From (2) and (3),

$$\begin{aligned} |l' - l| &= |l' - a_t + a_t - a_s + a_s - l|, \\ &\leq |a_t - l'| + |a_s - a_t| + |a_s - l|, \\ &< \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon, \\ &< \epsilon. \end{aligned}$$

$\therefore l' = l$. Thus $\{a_n\}$, having a unique limiting point, is convergent.

4.11. Theorems on Limits.

Theorem I: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences converging respectively to the limits a and b , then

- (i) $\text{Lt } (a_n + b_n) = a + b$,
- (ii) $\text{Lt } (a_n - b_n) = a - b$,
- (iii) $\text{Lt } (a_n \cdot b_n) = a \cdot b$,
- (iv) $\text{Lt } (a_n/b_n) = a/b$ unless $b = 0$.

(i) We have

$$|a_n - a| < \frac{1}{2}\epsilon \text{ for } n > \nu_1, \quad \dots \quad (1)$$

$$\text{and } |b_n - b| < \frac{1}{2}\epsilon \text{ for } n > \nu_2. \quad \dots \quad (2)$$

If ν be the greater of the two integers ν_1 and ν_2 , both (1) and (2) hold for $n > \nu$.

Since $(a_n + b_n) - (a + b) = (a_n - a) + (b_n - b)$, we have

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \\ &< \epsilon \end{aligned}$$

for $n > \nu$.

$$\therefore \text{Lt } (a_n + b_n) = a + b.$$

(ii) The proof is exactly similar to the above.

(iii) Let $a_n = a + \alpha$ and $b_n = b + \beta$, where α, β tend to 0 as $n \rightarrow \infty$.

$$\text{Then } a_n b_n - ab = (a + \alpha)(b + \beta) - ab = a\beta + \alpha(b + \beta).$$

Choose a positive number l greater than $|b + \beta|$. We have

$$|a_n b_n - ab| < |a| |\beta| + |\alpha| \cdot l.$$

By the definition of a limit we can choose m so that

$$|\alpha| = |a_n - a| < \frac{1}{2} \frac{\epsilon}{l} \text{ and } |\beta| = |b_n - b| < \frac{1}{2} \frac{\epsilon}{|a|}$$

provided $n > m$. Hence

$$|a_n b_n - ab| < \epsilon$$

for $n > m$; i.e., $\text{Lt } (a_n \cdot b_n) = a \cdot b$.

(iv) To prove this we first prove the following :

If the sequence $\{b_n\}$ tends to the limit b , then the sequence $\left\{\frac{1}{b_n}\right\}$ tends to the limit $\frac{1}{b}$, unless $b=0$.

Let $b_n = b + \beta$ where $\beta \rightarrow 0$ as $n \rightarrow \infty$.

Now $\frac{1}{b_n} - \frac{1}{b} = -\frac{\beta}{b(b+\beta)}$. If b is not zero, we can choose a positive number l less than $|b+\beta|$ and then

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| < \frac{|\beta|}{|b| \cdot l}.$$

Now we can choose m so that

$$|\beta| = |b_n - b| < |b| \cdot l \cdot \epsilon \text{ for } n > m.$$

Hence $\left| \frac{1}{b_n} - \frac{1}{b} \right| < \epsilon$ for $n > m$.

$$\therefore \text{Lt } \left(\frac{1}{b_n} \right) = \frac{1}{b}.$$

Now applying (iii) to the sequences $\{a_n\}$ and $\left\{\frac{1}{b_n}\right\}$, we get

$$\text{Lt } \left(\frac{a_n}{b_n} \right) = \frac{a}{b} \text{ unless } b=0.$$

4.12. Theorem II :

(1) If u_n is positive, and for all values of $n \geq m$ where m is a fixed integer, $u_{n+1} > k u_n$, where k is a constant greater than unity, then $u_n \rightarrow \infty$ as $n \rightarrow \infty$.

Rejecting the first m terms for which the condition does not hold and denoting the terms $u_{m+1}, u_{m+2}, \dots$ by $a_1, a_2, \dots$, it is clear that for all values of n , $a_{n+1} > k a_n$.

$$\therefore a_n > k a_{n-1} > k^2 a_{n-2} > \dots > k^{n-1} a_1.$$

Now $k > 1$, $\therefore a_n \rightarrow \infty$ as $n \rightarrow \infty$. It follows that $u_n \rightarrow \infty$ as $n \rightarrow \infty$.

(2) If u_n is positive and for all values of $n \geq m$ where m is a fixed

integer, and $u_{n+1} < ku_n$ where k is a positive constant less than unity, then $u_n \rightarrow 0$ as $n \rightarrow \infty$.

Proceeding as above, we have for all values of n , $a_{n+1} < ka_n$.

$$\therefore a_n < ka_{n-1} < k^2 a_{n-2} < \dots < k^{n-1} a_1.$$

As $k < 1$, $k^{n-1} \rightarrow 0$ as $n \rightarrow \infty$ and hence $a_n < k^{n-1} a_1 \rightarrow 0$ as $n \rightarrow \infty$.

It follows that $u_n \rightarrow 0$ as $n \rightarrow \infty$.

(3) If $u_n > 0$ and $\text{Lt} \left(\frac{u_{n+1}}{u_n} \right) = l$, then $u_n \rightarrow \infty$ if $l > 1$ and $u_n \rightarrow 0$ if $l < 1$.

If $l > 1$, we can choose k so that $l > k > 1$. Now since $\frac{u_{n+1}}{u_n} \rightarrow l$, we have

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \epsilon \text{ for } n \geq m,$$

$$\text{i.e., } u_n(l - \epsilon) < u_{n+1} < u_n(l + \epsilon).$$

As $l > 1$, ϵ can be chosen sufficiently small to make $l - \epsilon = k > 1$;

$\therefore u_{n+1} > ku_n$ for $n \geq m$; hence by (1), $u_n \rightarrow \infty$ as $n \rightarrow \infty$.

Again if $l < 1$, ϵ can be chosen sufficiently small to make

$$l + \epsilon = k < 1;$$

$\therefore u_{n+1} < ku_n$ for $n \geq m$ and so by (2), $u_n \rightarrow 0$ as $n \rightarrow \infty$.

4.13. An important limit :

$$\text{Lt}_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e.$$

First observe that in the above limit, $n \rightarrow \infty$ through positive integral values.

Let $a_n = \left(1 + \frac{1}{n} \right)^n$, where n is a positive integer. Then

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n} \right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{n^2} + \dots + \frac{n(n-1) \dots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n} \right) + \dots + \frac{1}{n!} \left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) \dots \left(1 - \frac{n-1}{n} \right) \\ &\quad \dots \quad (1) \end{aligned}$$

Each of the factors $\left(1 - \frac{1}{n}, \left(1 - \frac{2}{n}\right), \dots\right)$ is positive and the number of terms is $n+1$. Hence as n increases, a_n increases monotonically. Further as each of the above factors is less than 1, it follows that

$$a_n < 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}, \quad \dots \quad (2)$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}.$$

$$< 1 + 2\left(1 - \frac{1}{2^n}\right),$$

$$< 3.$$

$\therefore \{a_n\}$ being bounded above is convergent and converges to a limit λ , such that $2 < \lambda < 3$.

Consider now the sequence $\{b_n\}$ where

$$b_n = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}.$$

As above $b_n < 3$ and so $\{b_n\}$ is convergent and tends to a limit lying between 2 and 3; this limit is denoted by e .

From, (2), $a_n < b_n$; $\therefore \text{Lt } a_n \leq \text{Lt } b_n$,

$$\text{or, } \lambda \leq e.$$

Let m be any integer greater than n . Then, as in (1),

$$a_m = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{m}\right) + \dots + \frac{1}{n!} \left(1 - \frac{1}{m}\right) \left(1 - \frac{2}{m}\right) \dots \left(1 - \frac{n-1}{m}\right)$$

$$+ \dots + \frac{1}{m!} \left(1 - \frac{1}{m}\right) \left(1 - \frac{2}{m}\right) \dots \left(1 - \frac{m-1}{m}\right)$$

$$> 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{m}\right) + \dots + \frac{1}{n!} \left(1 - \frac{1}{m}\right) \left(1 - \frac{2}{m}\right) \dots \left(1 - \frac{n-1}{m}\right),$$

as several positive factors have been left out.

Keeping n fixed, let $m \rightarrow \infty$; then the right side $\rightarrow b_n$ and the left side to λ .

$\therefore \lambda \geq b_n$ for all integral values of n .

Hence $e = \text{Lt } b_n \leq \lambda$.

$$\dots \quad \dots \quad (4)$$

From (3) and (4), we get $\lambda = e$, that is,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Note. It can be proved that e is the incommensurable number 2.71828182...

4.14. Cauchy's limit theorems :

First theorem. If $\lim a_n = l$, then $\lim \frac{a_1 + a_2 + \dots + a_n}{n} = l$.

We write $a_n - l = b_n$ so that $\lim b_n = 0$. We have

$$\frac{a_1 + a_2 + \dots + a_n}{n} = l + \frac{b_1 + b_2 + \dots + b_n}{n};$$

so we have to prove that

$$\frac{b_1 + b_2 + \dots + b_n}{n} \rightarrow 0 \text{ when } b_n \rightarrow 0.$$

Let ϵ be any positive number, however small. Then there exists a positive integer m such that $|b_n| < \frac{\epsilon}{2}$ when $n \geq m$. We now write

$$\left| \frac{b_1 + b_2 + \dots + b_n}{n} \right| = \left| \frac{b_1 + b_2 + \dots + b_m}{n} + \frac{b_{m+1} + b_{m+2} + \dots + b_n}{n} \right|$$

Since $\{b_n\}$ is convergent, it is bounded and therefore there exists a positive number k such that $|b_n| < k$ for all n .

$$\begin{aligned} \therefore \left| \frac{b_1 + b_2 + \dots + b_n}{n} \right| &\leq \frac{|b_1| + |b_2| + \dots + |b_m|}{n} \\ &\quad + \frac{|b_{m+1}| + |b_{m+2}| + \dots + |b_n|}{n} \\ &\leq \frac{km}{n} + \frac{n-m}{n} \cdot \frac{\epsilon}{2} \\ &< \frac{km}{n} + \frac{\epsilon}{2}. \end{aligned}$$

Keeping m fixed, we see that $\frac{km}{n} < \frac{\epsilon}{2}$ if $n > \frac{2km}{\epsilon}$. If m' be an integer greater than $\frac{2km}{\epsilon}$, we see that $\frac{km}{n} < \frac{\epsilon}{2}$ if $n > m'$. If μ be the greater of m and m' , we have for every $n > \mu$,

$$\left| \frac{b_1 + b_2 + \dots + b_n}{n} \right| < \epsilon$$

which proves the theorem.

Second theorem. If $u_n > 0$ for all values of n and $\frac{u_{n+1}}{u_n} \rightarrow l (> 0)$

where l is a fixed number, then $u_n^{\frac{1}{n}} \rightarrow l$.

Because $l > 0$, we can choose a positive number $\epsilon < l$ and as small as we please.

Since $\frac{u_{n+1}}{u_n} \rightarrow l$, we can find m so that $\left| \frac{u_{n+1}}{u_n} - l \right| < \epsilon$ for $n \geq m$,

$$\text{i.e., } l - \epsilon < \frac{u_{m+1}}{u_m} < l + \epsilon,$$

$$l - \epsilon < \frac{u_{m+2}}{u_{m+1}} < l + \epsilon,$$

$$\dots \dots \dots$$

$$l - \epsilon < \frac{u_n}{u_{n-1}} < l + \epsilon;$$

and by multiplication,

$$(l - \epsilon)^{n-m} < \frac{u_n}{u_m} < (l + \epsilon)^{n-m}.$$

But $l - \epsilon > 0$, it follows that

$$(l - \epsilon)^n < \frac{u_n}{u_m} (l - \epsilon)^m < \frac{u_n}{u_m} l^m,$$

$$\text{and } (l + \epsilon)^n > \frac{u_n}{u_m} (l + \epsilon)^m > \frac{u_n}{u_m} l^m.$$

$$\therefore (l - \epsilon)^n < \frac{u_n}{u_m} l^m < (l + \epsilon)^n,$$

$$\text{and } \left(\frac{u_m}{l^m} \right)^{\frac{1}{n}} (l - \epsilon) < u_n < \left(\frac{u_m}{l^m} \right)^{\frac{1}{n}} (l + \epsilon).$$

Now $\frac{u_m}{l^m}$ is a fixed number, therefore

$$\left| \left(\frac{u_m}{l^m} \right)^{\frac{1}{n}} - 1 \right| \rightarrow 0 \text{ and hence } u_n^{\frac{1}{n}} \rightarrow l.$$

Note. The theorem can be proved to be true when $l \rightarrow +\infty$ or $-\infty$. The converse of the theorem is not true.

4.15. **Lemma.** Bernoulli's Inequality.

For every $n \geq 2$ and $p > -1$ ($p \neq 0$), $(1+p)^n > 1+np$.

When $n=2$, $(1+p)^2 = 1+2p+p^2 > 1+2p$.

Thus the inequality is true for $n=2$. Suppose it holds for $n=k > 2$.

Then $(1+p)^k > 1+kp$.

$\therefore (1+p)^k \cdot (1+p) > (1+kp)(1+p)$,

or, $(1+p)^{k+1} > 1+(k+1)p+kp^2$,
 $> 1+(k+1)p$.

So the inequality assumed true for $n=k$ is true for $n=k+1$. As it holds for $n=2$, it holds for $n > 2$.

Examples :

1. (a) To show that the sequence $\{x^n\}$ is convergent if, and only if, $-1 < x \leq 1$.

[(i) Let $x > 1$. We write $x=1+h$, so that $h > 0$.

$\therefore x^n = (1+h)^n > 1+nh$.

Let δ be a positive number however large. We have $1+nh > \delta$ if $n > \frac{\delta-1}{h}$; we see that $x^n > \delta$ for $n \geq m$ where m is an integer just greater than $\frac{\delta-1}{h}$. $\therefore \text{Lt } x^n = \infty$.

(ii) Let $x=1$; so $x^n=1$ for all values of n .

$\therefore \text{Lt } x^n = 1$.

(iii) Let $0 < x < 1$. We write $x = \frac{1}{1+h}$, so that $h > 0$. We have

$$0 < x^n = \frac{1}{(1+h)^n} < \frac{1}{1+nh}.$$

Let ϵ be a positive number however small. Then $\frac{1}{1+nh} < \epsilon$ if $n > \left(\frac{1}{\epsilon} - 1\right)/h$ and if m be a positive integer $> \left(\frac{1}{\epsilon} - 1\right)/h$, we find $0 < x^n < \epsilon$, or $|x^n| < \epsilon$ for $n \geq m$, so that $\text{Lt } x^n = 0$.

(iv) Let $x=0$; clearly $\text{Lt } x^n = 0$.

(v) Let $-1 < x < 0$; we write $x = -\alpha$, so that $0 < \alpha < 1$ and we have $|x^n| = \alpha^n$. It follows from (iii) that $\text{Lt } x^n = 0$.

(vi) If $x = -1$, obviously $\{x^n\}$ oscillates finitely.

(vii) When $x < -1$, we write $x = -\alpha$ so that $\alpha > 1$. As $n \rightarrow \infty$, x^n takes values, both positive and negative, greater than any assigned number. Hence $\{x^n\}$ oscillates infinitely.]

(b) To show that $\left\{x^{\frac{1}{n}}\right\}$ converges to 1 when x is a fixed positive number and n a positive integer.

[(i) For $x = 1$, the result is obvious.

(ii) Let $x > 1$, then $x^{\frac{1}{n}} > 1$. Put $x^{\frac{1}{n}} = 1 + h$ ($h > 0$);

then $x = (1 + h)^n > 1 + nh > nh$; $\therefore h < \frac{x}{n}$ and so $0 < h < \frac{x}{n}$.

As $n \rightarrow \infty$, $\frac{x}{n} \rightarrow 0$ and $\therefore h \rightarrow 0$. Thus $\text{Lt } x^{\frac{1}{n}} = 1$ as $n \rightarrow \infty$.

(iii) Let $x < 1$; we write $x = \frac{1}{y}$, so that $y > 1$.

$$\therefore x^{\frac{1}{n}} = \frac{1}{y^{\frac{1}{n}}} \rightarrow 1 \text{ as } n \rightarrow \infty.]$$

(c) $\text{Lt } n^{\frac{1}{n}} = 1$ when $n \rightarrow \infty$ (n a positive integer).

(d) $\text{Lt } (nx^n) = 0$ as $n \rightarrow \infty$ and $|x| < 1$, n being a positive integer.

(e) $\text{Lt } \frac{x^n}{n} = \infty$ as $n \rightarrow \infty$ where $x > 1$.

(f) $\text{Lt } \frac{x^n}{n} = 0$, if $|x| \leq 1$.

(g) $\text{Lt } \frac{x^n}{n!} = 0$ as $n \rightarrow \infty$.

(h) $\text{Lt } \frac{m(m-1) \dots (m-n+1)}{n!} x^n = 0$ if $|x| < 1$.

[See Hardy, Pure Mathematics, 10th edition, pp. 141, 142.]

2. If $\{u_n\}$ converges to a and $\text{Lt } (u_n - v_n) = 0$, then $\{v_n\}$ also converges to a .

[Since $\text{Lt } u_n = a$, $|u_n - a| < \frac{\epsilon}{2}$ for $n > v_1$.

Also $\text{Lt } (u_n - v_n) = 0$, $|u_n - v_n| < \frac{\epsilon}{2}$ for $n > v_2$.

Now $|v_n - a| = |v_n - u_n + u_n - a| \leq |v_n - u_n| + |u_n - a|$
 $< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $n > v$ where v is the greater of v_1 and v_2 .

$\therefore \{v_n\}$ converges to a .]

3. $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are three sequences such that

(i) $a_n < b_n < c_n$ for every value of n , ... (1)

(ii) $\text{Lt } a_n = \text{Lt } c_n = l$;

then $\text{Lt } b_n = l$.

[Let ϵ be any positive number. There exists a positive integer m such that for every value of $n \geq m$

$$l - \epsilon < a_n < l + \epsilon \quad \dots \quad (2)$$

$$l - \epsilon < c_n < l + \epsilon \quad \dots \quad (3)$$

From (1), (2), (3), it follows that

$$l - \epsilon < b_n < l + \epsilon, \text{ for } n \geq m.$$

Hence $\text{Lt } b_n = l$.]

4. Apply Cauchy's First Limit theorem to prove that

$$(i) \quad \text{Lt}_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}{n} = 0;$$

$$(ii) \quad \text{Lt}_{n \rightarrow \infty} \frac{1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}}{n} = 0;$$

$$(iii) \quad n^{\frac{1}{n}} = \left(1 \cdot \frac{2}{1} \cdot \frac{3}{2} \dots \frac{n}{n-1}\right)^{\frac{1}{n}} \rightarrow 1 \text{ as } n \rightarrow \infty;$$

$$(iv) \quad \text{Lt}_{n \rightarrow \infty} \frac{1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + \dots + n^{\frac{1}{n}}}{n} = 1;$$

(v) If $y_1, y_2, \dots, y_n$ and η be all positive and if $y_n \rightarrow \eta$, then

$$u_n = (y_1 y_2 \dots y_n)^{\frac{1}{n}} \rightarrow \eta.$$

(vi) Find the limit as $n \rightarrow \infty$ of the sequence

$$\left\{ \left(\frac{2}{1} \right)^1 \left(\frac{3}{2} \right)^2 \left(\frac{4}{3} \right)^3 \dots \left(\frac{n+1}{n} \right)^n \right\}^{\frac{1}{n}}.$$

5. Apply Cauchy's Second Limit theorem to establish :

$$(i) \quad \text{Lt } \frac{n}{(n!)^{\frac{1}{n}}} = e.$$

$$(ii) \quad \text{Lt } \left[\frac{(2n)!}{(n!)^2} \right]^{\frac{1}{n}} = 4. \quad (C. H. 1961)$$

6. If $v_n = \frac{u_1 + u_2 + \dots + u_n}{n}$, then $\{u_n\}$ and $\{v_n\}$ are both increasing or both decreasing.

7. Prove that when $n \rightarrow \infty$,

$$(i) \quad \text{Lt } \left[\frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \dots + \frac{1}{\sqrt{(n^2+n)}} \right] = 1;$$

$$(ii) \quad \text{Lt } \left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2} \right] = 0;$$

$$(iii) \quad \text{Lt } \left[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{(n+1)}} + \dots + \frac{1}{\sqrt{(2n)}} \right] = \infty;$$

$$(iv) \quad \text{Lt } \left[\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2} \right] = \frac{1}{2}.$$

8. If $x_{n+1} = \sqrt{k+x_n}$ and k and x_1 are positive, then the sequence $\{x_n\}$ is increasing or decreasing as x_1 is less or greater than α , the positive root of the equation $x^2 = x + k$; and in either case, $x_n \rightarrow \alpha$ as $n \rightarrow \infty$.

[We have $x_{n+1}^2 - x_n^2 = (k+x_n) - (k+x_{n-1}) = x_n - x_{n-1}$;

$\therefore x_{n+1} \geq x_n$ according as $x_n \geq x_{n-1}$. So the sequence is monotone, increasing if $x_1 < x_2$ and decreasing if $x_1 > x_2$.

The equation $x^2 - x - k = 0$ has two roots, one positive root α and one negative root $-\beta$.

$$\therefore x^2 - x - k = (x - \alpha)(x + \beta). \quad \dots \dots (1)$$

If $0 < x_1 < \alpha$, $x_1^2 - x_1 - k < 0$, or $x_1^2 < k + x_1$, i.e., $x_1 < \sqrt{k+x_1} = x_2$ and the sequence is increasing; if however $x_1 > \alpha$, then $x_1^2 - x_1 - k > 0$ which gives $x_1 > x_2$ and the sequence is decreasing.

From (1), $\alpha - \beta = 1$, and $\alpha\beta = k$.

If $x_1 < \alpha$, $\{x_n\}$ is increasing; $\therefore x_n < x_{n+1} = \sqrt{k+x_n}$,
or, $x_n^2 - x_n - k < 0$, or, $x_n^2 - (\alpha - \beta)x_n - \alpha\beta < 0$, i.e., $(x_n - \alpha)(x_n + \beta) < 0$, or, $x_n < \alpha$ for all values of n . Thus $\{x_n\}$ is monotone increasing and bounded above; it must therefore tend to a limit l .

Since $x_{n+1} = \sqrt{k+x_n}$, or $x_{n+1}^2 - x_n - k = 0$, taking limits $l^2 - l - k = 0$, or, l is the positive root of $x^2 - x - k = 0$.

The case $x_1 > \alpha$ may be dealt with similarly.]

9. Show that the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$ tends to a definite limit (stating clearly the theorem necessary to justify your argument) and find the limit. (C. H. 1960)

[Here $u_n = \sqrt{2+u_{n-1}}$ and $u_n < 2$ for all n . Required limit = 2.]

10. Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$ converges to 2. [$u_n < 2$ for all n .]

11. Prove that the sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \dots$ converges to the positive root of $x^2 - x - 7 = 0$.

[$u_n < 4$ for all n .]

12. If $\{x_n\}, \{y_n\}$ are two positive sequences such that $x_{n+1} = \frac{1}{2}(x_n + y_n)$ and $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$, show that both the sequences are monotonic, the former descending and the latter ascending. Show further that they tend to a common limit l , where $l^2 = x_1 y_1$.

[Since $A. M. \geq H. M.$, we have $\frac{x_n + y_n}{2} \geq \frac{2}{\frac{1}{x_n} + \frac{1}{y_n}}$,

i.e., $x_{n+1} \geq y_{n+1}$ for all n . Now $x_{n+1} = \frac{1}{2}(x_n + y_n) \leq \frac{1}{2}(x_n + x_n)$, i.e., $x_{n+1} \leq x_n$. Hence $\{x_n\}$ is monotone descending.

Again $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n} \leq \frac{1}{y_n} + \frac{1}{y_n}$, i.e., $y_n \leq y_{n+1}$; so $\{y_n\}$ is monotone ascending.

Since $x_n \geq y_n \geq y_1$, $\{x_n\}$ is bounded below and therefore tends to a limit l . Also $y_n \leq x_n \leq x_1$, $\{y_n\}$ is thus bounded above and tends to a limit m .

Now $x_{n+1} = \frac{1}{2}(x_n + y_n)$, taking limits, we find $l = m$.

Lastly $x_{n+1} = \frac{1}{2}(x_n + y_n)$ and $y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$.

$\therefore x_{n+1} y_{n+1} = x_n y_n = \dots = x_1 y_1$, taking limits $l^2 = x_1 y_1$.]

13. If $u_n = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots (2n)}$ and $v_n = \frac{3.5.7 \cdots (2n+1)}{2.4.6 \cdots (2n)}$,

prove that $u_n \rightarrow 0$, $v_n \rightarrow \infty$, and $\frac{1}{2} < u_n v_n < 1$. (C. H. 1963)

[$u_{n+1} - u_n < 0$ and $v_{n+1} - v_n > 0$; $\{u_n\}$ is monotone decreasing and $\{v_n\}$ is monotone increasing.

$$u_n = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots (2n)} < \frac{2.4.6 \cdots (2n)}{3.5.7 \cdots (2n+1)}, \therefore u_n^2 < \frac{1}{2n+1};$$

so $u_n \rightarrow 0$ as $n \rightarrow \infty$.

$$v_n = \frac{3.5.7 \cdots (2n+1)}{2.4.6 \cdots (2n)} > \frac{4.6.8 \cdots (2n+2)}{3.5.7 \cdots (2n+1)},$$

$\therefore v_n^2 > n+1$; thus $v_n \rightarrow \infty$ as $n \rightarrow \infty$.

$$u_n v_n < \frac{2.4.6 \cdots (2n)}{3.5.7 \cdots (2n+1)} v_n < 1;$$

$$u_n v_n > \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots (2n)} \cdot \frac{4.6.8 \cdots (2n+2)}{3.5.7 \cdots (2n+1)} > \frac{n+1}{2n+1} > \frac{1}{2}.]$$

14. Show that the sequence $\{y_n\}$ where $y_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is monotone descending. (C. H. 1958)

[$\{y_n\}$ is monotone descending if $y_{n-1} > y_n$, i.e., if

$$\left(1 + \frac{1}{n-1}\right)^n > \left(1 + \frac{1}{n}\right)^{n+1}; \text{ or, } \left(\frac{1 + \frac{1}{n-1}}{1 + \frac{1}{n}}\right)^n > 1 + \frac{1}{n}.$$

$$\text{Now } \left(\frac{1 + \frac{1}{n-1}}{1 + \frac{1}{n}}\right)^n = \left(\frac{n^2}{n^2-1}\right)^n = \left(1 + \frac{1}{n^2-1}\right)^n > 1 + \frac{n}{n^2-1}$$

(Art. 4'15)

$$> 1 + \frac{n}{n^2} = 1 + \frac{1}{n}. \text{ Hence }]$$

15. If $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$, and $y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \log n$, then $\{x_n\}$ is monotone decreasing and $\{y_n\}$ is monotone increasing and both converge to the same limit γ where $0 < \gamma < 1$. ($\gamma = 0.5772 \cdots$ and is known as **Euler's constant**)

[We have $e^x > 1+x$, $\therefore x > \log(1+x)$, or, $\frac{1}{n} > \log\left(1 + \frac{1}{n}\right)$.

$$\begin{aligned}\text{Again } \log \left(1 + \frac{1}{n}\right) &= -\log \frac{n}{n+1} = -\log \left(1 - \frac{1}{n+1}\right) \\ &= \frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots > \frac{1}{n+1}.\end{aligned}$$

$$\text{Thus } \frac{1}{n+1} < \log \left(1 + \frac{1}{n}\right) < \frac{1}{n} \quad \dots \quad (1)$$

Now $x_{n+1} - x_n = \frac{1}{n+1} - \log \left(1 + \frac{1}{n}\right) < 0$ [by (1)]; so $\{x_n\}$ is monotone decreasing.

Again $y_{n+1} - y_n = \frac{1}{n} - \log \left(1 + \frac{1}{n}\right) > 0$ [by (1)]; $\therefore \{y_n\}$ is monotone increasing.

In $\frac{1}{n} < \log(n+1) - \log n$, putting $n=1, 2, 3, \dots, n$ and adding, we get $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \log(n+1) > \log n$; so $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n > 0$.

As x_n is positive and $\{x_n\}$ is monotone decreasing, it converges to a limit $\gamma \geq 0$. Again $x_1 = 1$; $\therefore \gamma < 1$. Thus $0 \leq \gamma < 1$.

Now $x_n - y_n = \frac{1}{n}$ and hence $\text{Lt } (x_n - y_n) = 0$; $\therefore \{y_n\}$ converges to the same limit γ .

But $y_1 = 1 - \log 2 < 0.3$; $\therefore \gamma > 0.3$. Thus $0.3 < \gamma < 1$, or, $0 < \gamma < 1$.]

16. Prove rigorously that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ to infinity.
(C. H. 1962)

[From the above example,

$$\text{Lt } y_{n+1} = \text{Lt } \left\{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log(n+1)\right\} = \gamma.$$

$$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \log(n+1) + \gamma + \epsilon \text{ where } \epsilon \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\text{So } 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n} \quad \dots \quad (1)$$

$$= \log(2n+1) + \gamma + \epsilon' \text{ where } \epsilon' \rightarrow 0 \text{ as } n \rightarrow \infty. \quad \dots (2)$$

$$\begin{aligned}\text{Now (1) may be written } 2 \left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right) \\ = \log(n+1) + \gamma + \epsilon. \quad \dots (3)\end{aligned}$$

Subtracting (3) from (2), we get

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = \log \frac{2n+1}{1+n} + \epsilon' - \epsilon = \log \frac{2 + \frac{1}{n}}{1 + \frac{1}{n}} + \epsilon' - \epsilon.$$

As $n \rightarrow \infty$, we have

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{to infinity.}]$$

17. If $\{u_n\}$ and $\{v_n\}$ are two bounded sequences, prove that

$$(i) \quad \overline{\lim} (u_n + v_n) \leq \overline{\lim} u_n + \overline{\lim} v_n, \quad (C. H. 1960)$$

$$(ii) \quad \underline{\lim} (u_n + v_n) \geq \underline{\lim} u_n + \underline{\lim} v_n. \quad (C. H. 1964)$$

$$[(i) \quad \text{Let } \overline{\lim} u_n = G_1, \overline{\lim} v_n = G_2, \overline{\lim} (u_n + v_n) = G.]$$

If ϵ be any positive number, there exist positive integers m_1 and m_2 such that

$$u_n < G_1 + \frac{1}{2}\epsilon \text{ for } n \geq m_1, v_n < G_2 + \frac{1}{2}\epsilon \text{ for } n \geq m_2.$$

If m be the greater of m_1 and m_2 ,

$$u_n + v_n < G_1 + G_2 + \epsilon \text{ for } n \geq m.$$

Thus $G < G_1 + G_2 + \epsilon$.

As ϵ is arbitrary, $G \leq G_1 + G_2$,

$$\text{or } \overline{\lim} (u_n + v_n) \leq \overline{\lim} u_n + \overline{\lim} v_n.$$

$$(ii) \quad \text{Let } \underline{\lim} u_n = g_1, \underline{\lim} v_n = g_2, \underline{\lim} (u_n + v_n) = g.$$

ϵ being arbitrary, we can find positive integers m_1 and m_2 such that,

$$u_n > g_1 - \frac{1}{2}\epsilon \text{ for } n \geq m_1, v_n > g_2 - \frac{1}{2}\epsilon \text{ for } n \geq m_2.$$

$$\therefore u_n + v_n > g_1 + g_2 - \epsilon \text{ for } n > \max. (m_1, m_2).$$

So $g > g_1 + g_2 - \epsilon$.

Also as ϵ is arbitrary,

$$g \geq g_1 + g_2,$$

$$\text{or } \underline{\lim} (u_n + v_n) \geq \underline{\lim} u_n + \underline{\lim} v_n.]$$

18. If $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, show that $\{S_n\}$ is convergent.

(C. H. 1964)

$$[S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{(2n+1)(2n+2)} > 0.]$$

So $\{S_n\}$ is monotone increasing. Again for every n ,

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{n}{n+1} = 1 - \frac{1}{n+1} < 1.$$

Thus $\{S_n\}$ is bounded and therefore convergent.]

CHAPTER 5

FUNCTIONS

Of A Real Variable

5.1. Function of a real variable :

If x and y be two real variables so related that corresponding to every value of x in the interval $a \leq x \leq b$, y has got a definite value, y is called a function of x . x is called the independent variable and y the dependent variable, and the relation between x and y is symbolically represented by an equation of the form : $y = f(x)$, $y = F(x)$, $y = \phi(x)$, $y = \psi(x)$, ...

The interval (a, b) in which x varies is called the *domain* of the function, and the aggregate of values of y forms the *range* of the function.

Note 1. The following is the modern definition of a function. A function, represented by a symbol like f , is a rule or set of rules which associates every member of a set X of real numbers with the members of another set Y of real numbers ; in other words, f transforms a member of the set X into a member or members of the set Y . If f associates each member of the set X with one member only of the set Y , f is called a single-valued or uniform function ; whereas if f associates each member of the set X with more than one member of the set Y , f is a many-valued function. The member y of the set Y with which f associates the member x of the set X is called the value of the function for x and is represented by $y = f(x)$. The set X is called the *domain* of the function and the set Y is the *range* of the function.

Although f denotes a function and $f(x)$ a value of the function, it has been customary to denote the function itself by $f(x)$ and call it a function of x . We shall adopt this practice in what follows.

Note 2. In the days of Newton and Leibnitz, a variable y was called a function of x when it was connected to x by an equation involving symbols of mathematical operations. The definition given in Art. 5.1 has been evolved by a series of generalisations by Cauchy, Riemann, Dirichlet.

Note 3. When to each value of x in $a \leq x \leq b$ corresponds *one and only one* value of y , y is called a single-valued or uniform function of x . Dirichlet's definition of a function is : " y is a single-valued function of the variable x , in the continuous interval (a, b) when a definite value of y corresponds to each value of x such that $a \leq x \leq b$ no matter in what form correspondence is specified." Sometimes the expression or equation, which defines the function, gives two or more values of y for each value of x ; we may then consider the expression or equation, as defining two or more single-valued functions of x . Thus $y^2 = x$ can be looked upon as defining two distinct functions : $y = +\sqrt{x}$ and $y = -\sqrt{x}$.

Note 4. If y be a function of x , x may sometimes be looked upon as a function of y in virtue of the functional relation between x and y , the proper interval of y being taken into account. Usually the interval in which x varies will not be the same as the interval in which y varies.

Note 5. For y to be a function of x , to each value of x must correspond a definite value of y . However for the function $y = \frac{x^2 - 4}{x - 2}$, y has no definite value for $x = 2$, although for the function $y = x + 2$ obtained from the first by removing the common factor from numerator and denominator, y has the value 4 for $x = 2$. The first function is thus not defined for $x = 2$ and the interval in which it is defined is the aggregate of all real numbers with the exception of 2. The second function however is defined for all real values of x .

Note 6. It is to be borne in mind that for y to be a function of x , it is not essential that y should be expressible in terms of x by means of a mathematical formula.

Note 7. A function may be defined by different specifications or different laws in different parts of a given interval.

5'2. Different Classes of Functions.

A. Polynomials: A polynomial in x is a function of x of the form

$$a_0 x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m,$$

where $a_0, a_1, \dots, a_m$ are constants and m is a positive integer.

B. Rational Functions: A rational function is the quotient of one polynomial by another; thus if $P(x)$ and $Q(x)$ are polynomials, we denote the general rational function by

$$R(x) = \frac{P(x)}{Q(x)}.$$

(i) We usually suppose that $P(x)$ and $Q(x)$ have no common factors, all such factors, if any, being removed by division.

(ii) It should be observed that this removal of common factors, *does, as a rule, change the function*. The function $y = (x^2 - 4)/(x - 2)$ is an example to the point.

(iii) The constants occurring as co-efficients in a rational function need not be rational numbers. The use of the word 'rational' arises as follows. The rational function $P(x)/Q(x)$ may be generated from x by a finite number of operations upon x , including multiplication of x by itself or a constant, addition of terms thus obtained and division of one function, obtained by such multiplications and addi-

tions, by another. In so far as the variable x is concerned, the process is very much like that by which all rational numbers can be obtained from unity.

C. Algebraical Functions : y is an algebraical function of x , of degree m , if it is a root of an equation of degree m in y whose co-efficients are rational functions of x ; e.g.,

$$y^m + R_1 y^{m-1} + \dots + R_{m-1} y + R_m = 0.$$

Now this is not always solvable. If it is solvable and y is expressible in terms of x , we get what is called an *explicit* function of x . If not, the equation defines an *implicit* function of x . It should be noted that for a given value of x , the equation gives only a finite number of values of y .

D. Transcendental Functions : All functions which are not rational nor even algebraical are called *transcendental* functions. These include an infinite variety of classes of functions of varying degrees of simplicity and importance.

(a) The direct and inverse trigonometrical functions or the circular functions: These are the sine and cosine functions of elementary trigonometry, together with the functions derived from them.

(b) Other classes of transcendental functions: The most important among these are the exponential and logarithmic functions. Other functions of this class are the elliptic functions, Bessel's and Legendre's functions, the Gamma functions, etc.

5.3. Graphs of Functions.

(a) $y=x$; $y=x^3$. fig. 7)

Both are defined for all real values of x ; the first is a straight line and the second is the dotted curve in the diagram.

(b) $y=x^2$; $y=x^4$. (fig. 8)

Both are defined for all real values of x ; the first is a parabola and the second is the dotted curve in the diagram.

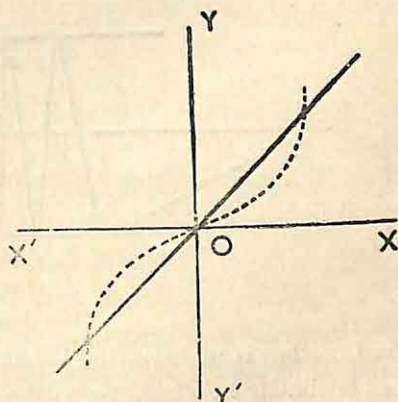


Fig. 7

$$(c) \left. \begin{array}{l} y = x^{\frac{1}{2}}, \\ \text{or } y^2 = x, \end{array} \right\}; \left. \begin{array}{l} y = x^{\frac{1}{3}}, \\ \text{or } y^3 = x, \end{array} \right\}; \left. \begin{array}{l} y = x^{\frac{1}{4}}, \\ \text{or } y^4 = x, \end{array} \right\}.$$

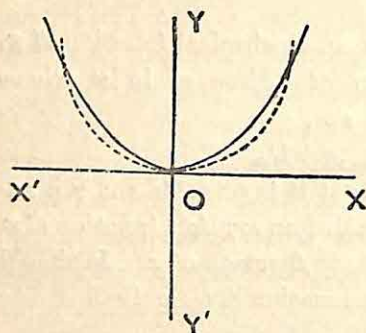


Fig. 8

On interchanging x and y , these reduce to functions considered in (a) and (b) and their graphs are reflections of the graphs of

$$y = x^2, y = x^3, y = x^4$$

on the line $y = x$.

$$(d) \quad y = x + 2; y = \frac{x^2 - 4}{x - 2}.$$

The graphs are straight lines; to the eye the graph of $y = x + 2$ is the same straight line as that of $y = \frac{x^2 - 4}{x - 2}$ but in the graph of the latter function, a point (the extremity of the ordinate corresponding to $x = 2$) is missing.

(e) $y = \sin\left(\frac{1}{x}\right)$. Here $y = 0$ when $x = \frac{1}{m\pi}$ where m is any integer. Again $y = 1$ when $x = \frac{1}{(2m + \frac{1}{2})\pi}$ and $y = -1$ when $x = \frac{1}{(2m - \frac{1}{2})\pi}$. The graph is entirely comprised between the lines $y = 1$ and $y = -1$. It oscillates up and down, the rapidity of the oscillations being greater and greater as x approaches 0. For $x = 0$,

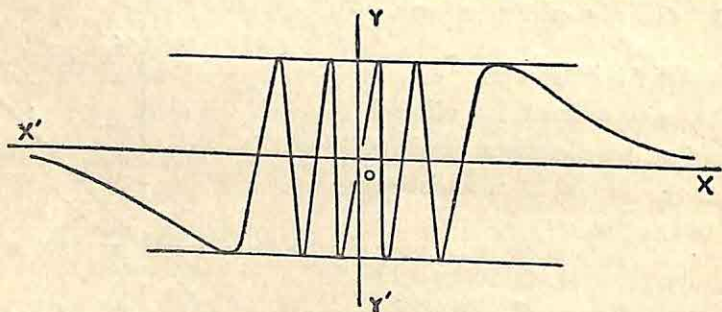


Fig. 9

the function is undefined. When x is large, y is small. The negative half is similar in character to the positive half. (fig. 9)

(f) $y = x \sin\left(\frac{1}{x}\right)$. The graph is comprised between the lines

$y=x$ and $y=-x$. For $x=0$, the function is undefined ; further as $x \rightarrow \infty$, $y \rightarrow 1$. (fig. 10)

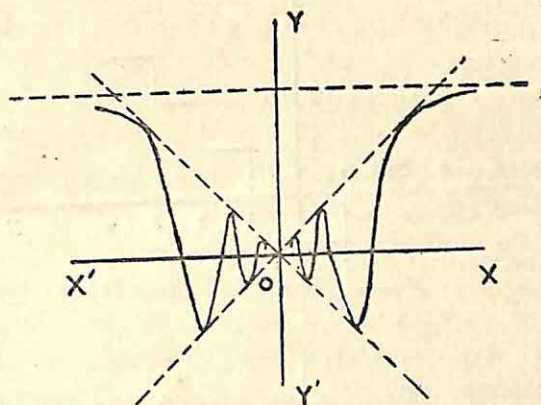


Fig. 10

(g) $y=e^x$. The function is positive for all real values of x . For $x=0$, $y=1$; as x increases, y increases monotonically ; if x is negative and increases in magnitude, y approaches 0. (fig. 11)

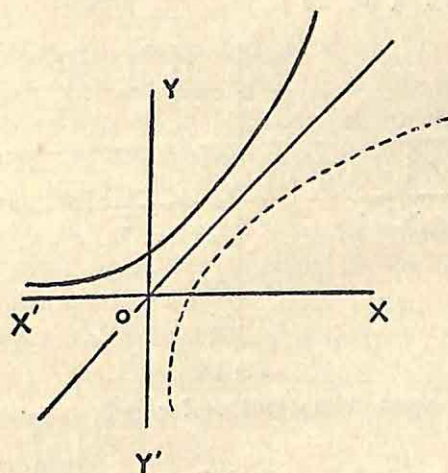


Fig. 11

(h) $y=\log x$. If $y=e^x$ interchanging x and y , we get $x=e^y$ which is the same as $y=\log x$. The graph of $y=\log x$ is the reflection of the graph of $y=e^x$ on $y=x$. It is the dotted curve in fig. 11.

(i) $y = [x]$, where $[x]$ denotes the greatest integer not greater than x

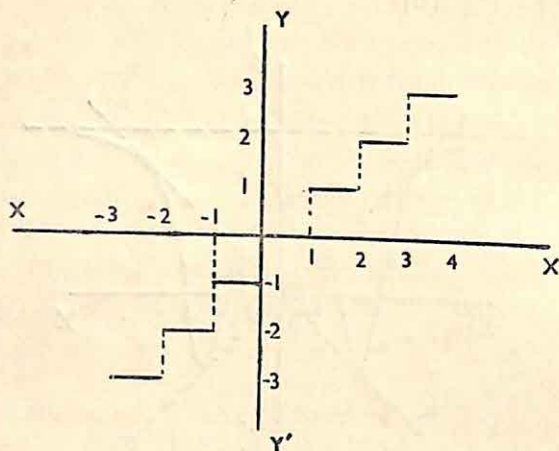


Fig. 12

than x . The graph is shown in the diagram (fig. 12) The left-hand end-points of the thick lines, but not the right-hand ones, belong to the graph.

(j) $y = x - [x]$. (fig. 13)

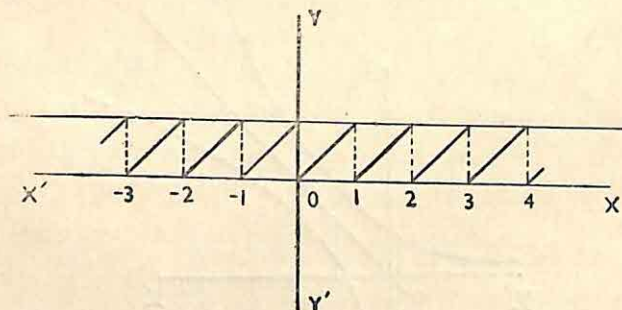


Fig 13.

(k) y = the largest prime factor of x . y is defined for all integral values of x . If

$$x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \dots$$

$$\text{then } y = 1, 2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 3, 13, \dots$$

The graph consists of isolated points.

5.4. Bounds of a Function : Bounded Function.

If a function $f(x)$ be defined for all values of x in an interval (a, b) , then for each value of x such that $a \leq x \leq b$, $f(x)$ has got a

definite value. If the aggregate of values of $f(x)$ has upper and lower bounds, M and m say, the function $f(x)$ is said to be a bounded function defined in the interval (a, b) and M and m are called the upper and lower bounds of the function in this interval. If the aggregate of values of $f(x)$ has only the upper bound or only the lower bound, the function is bounded above or bounded below as the case may be.

Note. A function is not necessarily bounded if for each value of x , $f(x)$ has got a definite value. Thus if $f(x)=0$ for $x=0$ and $f(x)=\frac{1}{x}$ for $x>0$, the function defined for positive values of x has no finite upper bound.

It may also be noted that the bounds of a function may not be values of the function for any values of x in the interval of definition; in other words, the bounds of a function may not be *attained*. Thus, if $f(x)=\sin x$ in $0<x<\frac{\pi}{2}$, the upper and lower bounds are 1 and 0 respectively, but $f(x)$ cannot attain these values in the open interval $(0, \frac{\pi}{2})$.

If the upper and lower bounds both exist for a function defined in a given domain, the difference between the two is called the *oscillation* of the function in the given domain.

5.5. Limit of a Function.

(a) Limit of a function of x as x tends to ' a '.

Let $f(x)$ be defined for all values of x in a neighbourhood of ' a ', except possibly for $x=a$. If $f(x)$ approaches a fixed number ' l ' as x approaches ' a ' in any manner whatsoever, $f(x)$ is said to tend to the limit ' l ' as x tends to ' a '.

More precisely : If corresponding to the positive number ϵ , chosen arbitrarily and as small as we please, we can find a positive number δ depending on ϵ , such that $|f(x)-l|<\epsilon$ when $0<|x-a|\leq\delta$, $f(x)$ is said to tend to the limit ' l ' as x tends to ' a '. This of course means that $f(x)$ lies between $l-\epsilon$ and $l+\epsilon$ for all those values of x , (except possibly $x=a$) which belong to the interval $(a-\delta, a+\delta)$. Symbolically we write

$$\text{Lt}_{x \rightarrow a} f(x) = l, \quad \text{or,} \quad \lim_{x \rightarrow a} f(x) = l.$$

Note. The part $0<|x-a|$ of the above inequality excludes the possibility $x=a$ and for the existence of the limit, it is immaterial whether $f(x)$ has or has not a value for $x=a$. The function may not *even* be defined for $x=a$. We may also replace the inequality $|x-a|\leq\delta$ by $|x-a|<\delta$.

(b) Right-handed limit.

If to every positive number ϵ corresponds a positive number δ , such that $|f(x) - l_1| < \epsilon$ when $a < x \leq a + \delta$, we say that $f(x) \rightarrow l_1$ as $x \rightarrow a$ through values of x greater than a and we write symbolically

$$\text{Lt}_{x \rightarrow a+0} f(x) = l_1, \quad \text{or,} \quad f(a+0) = l_1.$$

(c) Left-handed limit.

If to every positive number ϵ corresponds a positive number δ , such that $|f(x) - l_2| < \epsilon$ when $a - \delta \leq x < a$, we say that $f(x) \rightarrow l_2$ as $x \rightarrow a$ through values of x less than a . In symbols, we write

$$\text{Lt}_{x \rightarrow a-0} f(x) = l_2, \quad \text{or,} \quad f(a-0) = l_2.$$

(d) Unique limit.

If the right-hand limit l_1 be equal to the left-hand limit l_2 and the common value is l , then and then only

$$\text{Lt}_{x \rightarrow a} f(x) = l.$$

If $l_1 \neq l_2$, $\text{Lt}_{x \rightarrow a} f(x)$ does not exist.

$$(e) \quad \text{Lt}_{x \rightarrow \infty} f(x) = l; \quad \text{Lt}_{x \rightarrow \infty} f(x) = l.$$

If to every arbitrary positive number ϵ , there corresponds a positive number δ such that $|f(x) - l| < \epsilon$ when $x \geq \delta$, we say that $f(x) \rightarrow l$ as $x \rightarrow \infty$ and we write $\text{Lt}_{x \rightarrow \infty} f(x) = l$.

Similarly if to every arbitrary positive number ϵ corresponds a positive number δ such that $|f(x) - l| < \epsilon$ when $x \leq -\delta$, we say that $f(x) \rightarrow l$ as $x \rightarrow -\infty$ and we write $\text{Lt}_{x \rightarrow -\infty} f(x) = l$.

(f) Infinite limits.

It is easy to frame suitable definitions for

$$\text{Lt}_{x \rightarrow a} f(x) = \pm \infty, \quad \text{Lt}_{x \rightarrow \infty} f(x) = \pm \infty, \quad \text{Lt}_{x \rightarrow -\infty} f(x) = \pm \infty.$$

5'6. Criterion for the existence of a limit: General Principle of Convergence.

The necessary and sufficient condition that $f(x)$ may tend to a finite limit as x tends to a is that to every positive number ϵ , however small, there corresponds a positive number δ , such that

$$|f(x_2) - f(x_1)| < \epsilon$$

for every pair x_1, x_2 of values of x which satisfy

$$0 < |x_1 - a| \leq \delta, \quad 0 < |x_2 - a| \leq \delta,$$

i.e., for every pair x_1, x_2 of values of x , other than a which belong to the interval $(a - \delta, a + \delta)$.

The condition is *necessary*.

Let $f(x) \rightarrow l$ as $x \rightarrow a$. Let ϵ be any positive number. There exists a positive number δ such that

$$|f(x) - l| < \frac{\epsilon}{2} \text{ when } 0 < |x - a| \leq \delta.$$

If x_1, x_2 be any two numbers such that

$$0 < |x_1 - a| \leq \delta, \quad 0 < |x_2 - a| \leq \delta,$$

we have

$$|f(x_1) - l| < \frac{\epsilon}{2}, \quad |f(x_2) - l| < \frac{\epsilon}{2};$$

and so

$$\begin{aligned} |f(x_2) - f(x_1)| &= |f(x_2) - l + l - f(x_1)| \\ &\leq |f(x_2) - l| + |f(x_1) - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

The condition is *sufficient*.

Let $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_n, \dots$ be a monotone decreasing sequence of positive numbers which converges to 0 as n tends to infinity.

To each positive number ϵ_n corresponds a positive number δ_n such that $|f(x_2) - f(x_1)| < \epsilon_n$ when $0 < |x_1 - a| \leq \delta_n, 0 < |x_2 - a| \leq \delta_n$.

We thus obtain another sequence $\delta_1, \delta_2, \delta_3, \dots, \delta_n, \dots$ of positive numbers corresponding to the sequence $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_n, \dots$ and the sequence $\{\delta_n\}$ is obviously monotone decreasing. Writing $a + \delta_n$ for x_1 and x for x_2 we see that $|f(x) - f(a + \delta_n)| < \epsilon_n$ when

$0 < |x - a| \leq \delta_n$. Then for values of x other than a which belong to the interval $(a - \delta_n, a + \delta_n)$, $f(x)$ belongs to the open interval $[f(a + \delta_n) - \epsilon_n, f(a + \delta_n) + \epsilon_n]$ which we call A_n and whose length is $2\epsilon_n$. Thus we obtain a sequence of intervals $A_1, A_2, A_3, \dots, A_n, \dots$ such that each member of the sequence is contained in the preceding one. Also the length $2\epsilon_n$ of $A_n \rightarrow 0$ as $n \rightarrow \infty$.

There exists therefore one, and only one, point, say l , common to all the intervals of the sequence $\{A_n\}$. For let the intervals $A_1, A_2, A_3, \dots, A_n, \dots$ be $(a_1, b_1), (a_2, b_2), (a_3, b_3), \dots (a_n, b_n), \dots$. Then $a_1 < a_2 < a_3 < \dots < a_n < \dots$, and $b_1 > b_2 > b_3 > \dots > b_n > \dots$.

So $\{a_n\}$ is monotone increasing and bounded above by b_1 and $\{b_n\}$ is monotone decreasing and bounded below by a_1 . $\therefore$ both $\{a_n\}$ and $\{b_n\}$ tend to limits λ and μ say. Now $|\lambda - \mu| = \text{Lt } |a_n - b_n| = 0$.

$$\therefore \lambda = \mu = l \text{ say.}$$

Let ϵ be any positive number. We choose n so large that $2\epsilon_n < \epsilon$. We have, since l is contained in A_n ,

$$f(a + \delta_n) - \epsilon_n \leq l \leq f(a + \delta_n) + \epsilon_n, \quad \dots \quad \dots (1)$$

$$\text{also } f(a + \delta_n) - \epsilon_n < f(x) < f(a + \delta_n) + \epsilon_n, \quad \dots \quad \dots (2)$$

when $0 < |x - a| \leq \delta_n$.

From (1) we get $|l - f(a + \delta_n)| \leq \epsilon_n$

and from (2), $|f(x) - f(a + \delta_n)| < \epsilon_n$.

$$\begin{aligned} \therefore |f(x) - l| &= |f(x) - f(a + \delta_n) + f(a + \delta_n) - l| \\ &\leq |f(x) - f(a + \delta_n)| + |f(a + \delta_n) - l| \\ &< \epsilon_n + \epsilon_n \\ &< 2\epsilon_n \\ &< \epsilon, \end{aligned}$$

when $0 < |x - a| \leq \delta_n$.

Hence $f(x) \rightarrow l$ as $x \rightarrow a$.

5.7. Illustrations.

(a) Show that $\text{Lt}_{x \rightarrow 0} \cos \left(\frac{1}{x} \right)$ does not exist.

If the limit is to exist, corresponding to any ϵ , we must be able to find δ such that x_1 and x_2 being two values of x satisfying $|x| < \delta$, we must have $\left| \cos \left(\frac{1}{x_1} \right) - \cos \left(\frac{1}{x_2} \right) \right| < \epsilon$.

Now let $x_1 = \frac{1}{2n\pi}$ and $x_2 = \frac{1}{(2n+1)\pi}$, n being a positive integer ;
 whatever δ we may choose, we can make $\frac{1}{2n\pi}$ and $\frac{1}{(2n+1)\pi}$;
i.e., x_1 and x_2 less than δ by taking n sufficiently large. But $\left| \cos \left(\frac{1}{x_1} \right) - \cos \left(\frac{1}{x_2} \right) \right| = \left| \cos 2n\pi - \cos (2n+1)\pi \right| = 2$ and this is not less than any arbitrarily chosen ϵ . Hence the limit does not exist.

(b) Show that $\lim_{x \rightarrow 0} \frac{1}{2+e^{\frac{1}{x}}}$ does not exist.

If the limit is to exist, we must be able to find δ such that when $|x_1| \leq \delta$ and $|x_2| \leq \delta$,

$$\left| \frac{1}{2+e^{\frac{1}{x_1}}} - \frac{1}{2+e^{\frac{1}{x_2}}} \right| < \epsilon.$$

Now take $x_1 = -\frac{1}{n}$ and $x_2 = \frac{1}{n}$. By taking n sufficiently large,

$|x_1|$ and $|x_2|$ can be made less than any δ . Now

$$\begin{aligned} \left| \frac{1}{2+e^{\frac{1}{x_1}}} - \frac{1}{2+e^{\frac{1}{x_2}}} \right| &= \left| \frac{1}{2+e^{-n}} - \frac{1}{2+e^n} \right| \\ &> \left| \frac{1}{2+e^{-1}} - \frac{1}{2+e} \right| > \text{a finite quantity,} \end{aligned}$$

and so cannot be made less than any arbitrary ϵ . Hence the limit does not exist.

Note. $\lim_{x \rightarrow 0} x \cos \left(\frac{1}{x} \right)$ exists and $= 0$.

$$\text{Also } \lim_{x \rightarrow 0+0} \frac{1}{2+e^{\frac{1}{x}}} = 0; \quad \lim_{x \rightarrow 0-0} \frac{1}{2+e^{\frac{1}{x}}} = \frac{1}{2}.$$

5.8. Theorems on limits.

If $f(x)$ and $\phi(x)$ be two functions such that when $x \rightarrow a$, $\lim f(x) = l$ and $\lim \phi(x) = m$, then

- (i) $\lim \{f(x) \pm \phi(x)\} = \lim f(x) \pm \lim \phi(x) = l + m$;
- (ii) $\lim \{f(x) \cdot \phi(x)\} = \lim f(x) \cdot \lim \phi(x) = l \cdot m$;
- (iii) $\lim \{f(x)/\phi(x)\} = \lim f(x)/\lim \phi(x) = l/m$, provided $m \neq 0$.

(i) Since $f(x)$ and $\phi(x)$ tend respectively to l and m as $x \rightarrow a$, we have

$$\begin{aligned} |f(x) - l| &< \frac{1}{2}\epsilon \text{ for } 0 < |x - a| \leq \delta_1, \\ |\phi(x) - m| &< \frac{1}{2}\epsilon \text{ for } 0 < |x - a| \leq \delta_2. \end{aligned}$$

If δ be the smaller of δ_1 and δ_2 , both inequalities hold for $0 < |x - a| \leq \delta$.

$$\begin{aligned} \text{Now } |f(x) + \phi(x) - (l + m)| &\leq |f(x) - l| + |\phi(x) - m|, \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon, \\ &< \epsilon, \end{aligned}$$

for $0 < |x - a| \leq \delta$. Hence $f(x) + \phi(x) \rightarrow l + m$ as $x \rightarrow a$.

It can be shown similarly that $f(x) - \phi(x) \rightarrow l - m$ as $x \rightarrow a$.

$$\begin{aligned} \text{(ii) We have } |f(x) - l| &< \epsilon_1 \text{ for } 0 < |x - a| \leq \delta_1, \\ |\phi(x) - m| &< \epsilon_1 \text{ for } 0 < |x - a| \leq \delta_2. \end{aligned}$$

Both hold for $0 < |x - a| < \delta$, where δ is the smaller of δ_1 and δ_2 .

$$\text{Now } f(x) \cdot \phi(x) - lm = \{f(x) - l\} \cdot \{\phi(x) - m\} + l\{\phi(x) - m\} + m\{f(x) - l\}.$$

$$\begin{aligned} \text{Hence } |f(x) \cdot \phi(x) - lm| &\leq |f(x) - l| \cdot |\phi(x) - m| \\ &\quad + |l| \cdot |\phi(x) - m| + |m| \cdot |f(x) - l|, \\ &< \epsilon_1^2 + |l| \cdot \epsilon_1 + |m| \cdot \epsilon_1 \\ &< \epsilon_1 (\epsilon_1 + |l| + |m|) \\ &< \epsilon_1 (1 + |l| + |m|) \\ &< \epsilon, \text{ where } \epsilon_1 = \epsilon / (1 + |l| + |m|), \end{aligned}$$

for $0 < |x - a| \leq \delta$.

Thus $f(x) \cdot \phi(x) \rightarrow lm$ as $x \rightarrow a$.

(iii) Since $m \neq 0$, let $|m| = 2\eta$. Now $\phi(x) \rightarrow m$ as $x \rightarrow a$, so we have $|\phi(x) - m| < \eta$ for $0 < |x - a| \leq \delta_1$.

$$\text{Again } |m| - |\phi(x)| \leq |\phi(x) - m| < \eta,$$

$$\therefore |\phi(x)| > |m| - \eta = 2\eta - \eta = \eta, \text{ for } 0 < |x - a| \leq \delta_1 \quad \dots \quad (1)$$

$$\text{Also } |f(x) - l| < \epsilon_1, \text{ for } 0 < |x - a| \leq \delta_2, \quad \dots \quad (2)$$

$$|\phi(x) - m| < \epsilon_1, \text{ for } 0 < |x - a| \leq \delta_3. \quad \dots \quad (3)$$

If δ be the smallest of $\delta_1, \delta_2, \delta_3$, (1), (2), (3) all hold for $0 < |x - a| \leq \delta$.

$$\begin{aligned} \text{Now, } \left| \frac{f(x)}{\phi(x)} - \frac{l}{m} \right| &= \frac{|mf(x) - l\phi(x)|}{|m| \cdot |\phi(x)|} = \frac{|m\{f(x) - l\} - l\{\phi(x) - m\}|}{|m| \cdot |\phi(x)|} \\ &\leq \frac{|m| \cdot |f(x) - l| + |l| \cdot |\phi(x) - m|}{|m| \cdot |\phi(x)|} \end{aligned}$$

$$< \frac{\epsilon_1 (|l| + |m|)}{2\eta^2}$$

$$< \epsilon, \text{ where } \epsilon_1 = \frac{2\eta^2 \epsilon}{|l| + |m|}$$

$$\text{for } 0 < |x-a| \leq \delta.$$

$$\text{Thus } \lim_{x \rightarrow a} \frac{f(x)}{\phi(x)} = \frac{l}{m}, \text{ if } m \neq 0.$$

(iv) The limit of a function of a function.

$$\text{If } \lim_{x \rightarrow a} \phi(x) = b \text{ and } \lim_{\phi(x) \rightarrow b} f[\phi(x)] = f(b),$$

$$\text{i.e., putting } \phi(x) = u, \lim_{u \rightarrow b} f(u) = f(b), \text{ then}$$

$$\lim_{x \rightarrow a} f[\phi(x)] = f \left[\lim_{x \rightarrow a} \phi(x) \right].$$

We are given that $\lim_{u \rightarrow b} f(u) = f(b)$; therefore to the arbitrary positive number ϵ , there corresponds a positive number δ_1 such that

$$|f(u) - f(b)| < \epsilon, \text{ when } 0 < |u - b| \leq \delta_1, \\ \text{i.e., } |f[\phi(x)] - f(b)| < \epsilon, \text{ when } 0 < |\phi(x) - b| \leq \delta_1, \dots (1)$$

Again since $\lim_{x \rightarrow a} \phi(x) = b$, to the positive number δ_1 , there corresponds a positive number δ such that

$$|\phi(x) - b| < \delta_1, \text{ when } 0 < |x - a| \leq \delta. \dots (2)$$

Combining (1) and (2), we find that to the arbitrary positive number ϵ corresponds a positive number δ such that

$$|f[\phi(x)] - f(b)| < \epsilon, \text{ when } 0 < |x - a| \leq \delta.$$

$$\text{Thus } \lim_{x \rightarrow a} f[\phi(x)] = f(b) = f \left[\lim_{x \rightarrow a} \phi(x) \right]$$

5.9. Some useful limits.

$$(a) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{z \rightarrow 0} \left(1 + z\right)^{\frac{1}{z}} = e.$$

(i) If x be any real number greater than 1, we can always find two consecutive integers, n and $n+1$ such that

$$n \leq x < n+1;$$

$$\therefore 1 + \frac{1}{n} \geq 1 + \frac{1}{x} > 1 + \frac{1}{n+1},$$

$$\text{or, } \left(1 + \frac{1}{n}\right)^{n+1} \geq \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n+1}\right)^n.$$

As $x \rightarrow \infty$, n also $\rightarrow \infty$, we have

$$\text{Lt } \left(1 + \frac{1}{n}\right)^{n+1} = \text{Lt } \left(1 + \frac{1}{n}\right)^n \cdot \text{Lt } \left(1 + \frac{1}{n}\right) = e \cdot 1 = e;$$

$$\text{Lt } \left(1 + \frac{1}{n+1}\right)^n = \text{Lt } \left(1 + \frac{1}{n+1}\right)^{n+1} / \text{Lt } \left(1 + \frac{1}{n+1}\right) = \frac{e}{1} = e.$$

$$\text{Hence } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

(ii) Let $x = -y$ so that $y \rightarrow \infty$ as $x \rightarrow -\infty$. We have

$$\left(1 + \frac{1}{x}\right)^x = \left(1 - \frac{1}{y}\right)^{-y} = \left(1 + \frac{1}{y-1}\right)^{y-1} \cdot \left(1 + \frac{1}{y-1}\right)$$

$$\therefore \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x = e \cdot 1 = e.$$

(iii) Putting $z = \frac{1}{x}$, so that $z \rightarrow (0+0)$ or $(0-0)$ according as $x \rightarrow +\infty$ or $-\infty$, we get

$$\lim_{z \rightarrow 0} (1+z)^{\frac{1}{z}} = e.$$

$$(b) \quad \lim_{x \rightarrow 0} \frac{1}{x} \log(1+x) = 1.$$

$$(c) \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$$

$$(d) \quad \lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x} = n.$$

$$(e) \quad \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

$$(f) \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

5.10. Monotonic functions.

The interval $(a-\epsilon, a+\epsilon)$ containing a , where ϵ is a small positive number, is called a neighbourhood of a . If x_1 and x_2 are values of x such that $a-\epsilon < x_1 < x_2 < a+\epsilon$ and if $f(x_1) \leq f(x_2)$, $f(x)$ is said to be monotonic increasing in the neighbourhood of a ; again if $f(x_1) \geq f(x_2)$, $f(x)$ is monotonic decreasing in the neighbour-

hood of a . If a function is monotonic increasing or monotonic decreasing in the neighbourhood of every point of a given interval, it is said to be monotonic increasing or monotonic decreasing in that interval.

As regards the existence of limits, the properties of monotonic functions are similar to those of monotonic sequences and may be similarly proved.

We can easily prove the following results :

(i) If $f(x)$ is a monotonic increasing function in (a, ∞) and there exists a number k , such that $f(x) \leq k$, when $x \geq a$, then

$$\text{Lt}_{x \rightarrow \infty} f(x) \text{ exists and is } \leq k.$$

(ii) If $f(x)$ is monotonic decreasing in $(a, -\infty)$ and there exists a number k , such that $f(x) \geq k$, when $x \leq a$, then

$$\text{Lt}_{x \rightarrow -\infty} f(x) \text{ exists and is } \geq k.$$

Examples.

1. Use the definition of a limit to prove that if $\text{Lt}_{x \rightarrow a} f(x) = l$ and

k is a finite constant, then

$$\text{Lt}_{x \rightarrow a} \{kf(x)\} = kl.$$

2. $f(x)$, $\phi(x)$, $\psi(x)$ are three functions such that for all values of x (except possibly $x=a$) which lie in a certain neighbourhood of a ,

$$f(x) \leq \phi(x) \leq \psi(x)$$

and $\text{Lt}_{x \rightarrow a} f(x) = \text{Lt}_{x \rightarrow a} \psi(x) = l$ as $x \rightarrow a$.

Show that $\text{Lt}_{x \rightarrow a} \phi(x) = l$ as $x \rightarrow a$.

[Suppose $f(x) \leq \phi(x) \leq \psi(x)$, when $0 < |x-a| < \delta_1$. From the given limits we have, by definition,

$$l - \epsilon < f(x) < l + \epsilon, \text{ when } 0 < |x-a| < \delta_2,$$

$$l - \epsilon < \psi(x) < l + \epsilon, \text{ when } 0 < |x-a| < \delta_3.$$

If $\delta = \min. (\delta_1, \delta_2, \delta_3)$, all three inequalities hold when $0 < |x-a| < \delta$. Under these conditions,

$$l - \epsilon < f(x) \leq \phi(x),$$

and $l + \epsilon > \psi(x) \geq \phi(x)$;

i.e., $l - \epsilon < \phi(x) < l + \epsilon$ when $0 < |x - a| < \delta$,

or, $\text{Lt}_{x \rightarrow a} \phi(x) = l$.]

$x \rightarrow a$

3. If $\text{Lt}_{x \rightarrow a} f(x) = l$ and $\text{Lt}_{x \rightarrow a} \phi(x) = m$ and if $f(x) < \phi(x)$ in a

certain neighbourhood of a (except possibly $x = a$), then $l \leq m$.

[Suppose $f(x) < \phi(x)$ when $0 < |x - a| < \delta_1$,

i.e., in $a - \delta_1 < x < a + \delta_1$, ($x \neq a$). ... (1)

If possible, suppose $l > m$. Let us choose $\epsilon = \frac{1}{2}(l - m)$, a positive number.

Since $\text{Lt}_{x \rightarrow a} f(x) = l$, $\therefore |f(x) - l| < \epsilon$,

when $0 < |x - a| < \delta_2$,

i.e., $l - \epsilon < f(x) < l + \epsilon$, or, $l - \frac{1}{2}(l - m) < f(x)$,

i.e., $\frac{1}{2}(l + m) < f(x)$ (2)

Again since $\text{Lt}_{x \rightarrow a} \phi(x) = m$, $m - \epsilon < \phi(x) < m + \epsilon$,

when $0 < |x - a| < \delta_3$,

i.e., $\phi(x) < m + \frac{1}{2}(l - m)$, or, $\phi(x) < \frac{1}{2}(l + m)$ (3)

If $\delta = \min.(\delta_1, \delta_2, \delta_3)$, (1), (2), (3) hold for $0 < |x - a| < \delta$.

From (2) and (3), $f(x) > \frac{1}{2}(l + m) > \phi(x)$, i.e., $f(x) > \phi(x)$ in $0 < |x - a| < \delta$, which contradicts our hypothesis $f(x) < \phi(x)$. Hence our assumption $l > m$ is incorrect. $\therefore l \leq m$.]

4. Evaluate :

(i) $\text{Lt}_{n \rightarrow \infty} \frac{x^n}{x^n + 1}$; (ii) $\text{Lt}_{n \rightarrow \infty} \frac{x^n - 1}{x^n + 1}$; (iii) $\text{Lt}_{n \rightarrow \infty} \sin n\pi x$.

5. If $\phi(x) = \text{Lt}_{n \rightarrow \infty} \frac{x^n f(x) + g(x)}{x^n + 1}$, prove that

$\phi(x) = f(x)$, $g(x)$, $\frac{1}{2}\{f(x) + g(x)\}$, or undefined, according as $|x| > 1$, < 1 , $= 1$, or -1 .

6. If $f(x) = \text{Lt}_{n \rightarrow \infty} \frac{1}{1 + x^{2n}}$, show that $f(x) = 1, \frac{1}{2}, 0$ according

as $|x| < 1, = 1$, or > 1 .

7. If $f(x) = \lim_{n \rightarrow \infty} \frac{1}{1 + n \sin^2 \pi x}$ then $f(x) = 1$, when x is an integer and $f(x) = 0$ otherwise.

8. If $f(x) = \lim_{n \rightarrow \infty} \frac{2}{\pi} \arctan nx$, $f(x) = 1, 0$ or -1 according as $x > 0, = 0$, or < 0 .

[Note. A function which is 1, 0 or -1, according as $x > 0, = 0$, or < 0 , is called the *signum* function of x and written $\operatorname{sgn} x$. Hence

$$\operatorname{sgn} x = \lim_{n \rightarrow \infty} \frac{2}{\pi} \arctan nx.]$$

9. Prove that $\lim_{x \rightarrow 0} \tan^{-1}(a/x^2) = -\frac{\pi}{2}, 0$, or $\frac{\pi}{2}$ according as a is negative, zero or positive.

10. Draw the graphs of the functions :

$$(i) \quad f(x) = \lim_{t \rightarrow 0} \left(\frac{2x}{\pi} \tan^{-1} \frac{x}{t^2} \right);$$

$$(ii) \quad f(x) = \lim_{t \rightarrow 0} \left(\frac{2x^2}{\pi} \tan^{-1} \frac{x}{t^2} \right).$$

CHAPTER 6

CONTINUITY

6.1. Continuity at a point. Cauchy's definition.

A function $f(x)$ is said to be *continuous* at the point $x=c$ if (i) $f(c)$ exists, (ii) $f(x)$ exists at all points in the immediate neighbourhood of c and (iii) $f(x)$ tends to the limit $f(c)$ as x tends to c in any manner. Symbolically, $f(x)$ is continuous at $x=c$ if

$$f(c-0)=f(c)=f(c+0). \quad \dots \quad (1)$$

In the language of limits, $f(x)$ is continuous at $x=c$ if corresponding to any arbitrary positive number ϵ , we can find a positive number δ such that

$$|f(x)-f(c)| < \epsilon, \text{ when } 0 \leq |x-c| \leq \delta. \dots (2)$$

Note. The above definition applies to the case where c is an interior point of an interval (a, b) . For continuity at the lower end-point a , we must have $f(a)=f(a+0)$ and at the upper end-point b , $f(b)=f(b-0)$.

6.2. Continuity in an interval.

If $f(x)$ be continuous at all points of an interval (a, b) , it is said to be continuous in the interval (a, b) . If the interval is open, $f(x)$ must be continuous at all interior points of the interval; if it is closed, $f(x)$ must be continuous at the end-points as well.

6.3. Discontinuity.

If one or more of the conditions (1) be not satisfied, the function is *discontinuous* at $x=c$. This may happen in any one of the following ways :

I. (i) $f(c-0)=f(c+0) \neq f(c)$, i.e., $\lim_{x \rightarrow c} f(x)$ exists but is different from $f(c)$. In such a case, the discontinuity is said to be of the *first kind*.

Exs. (a) $f(x)=[1-(x-c)^2]$ at $x=c$ where $[x]$ denotes the greatest integer not greater than x .

$$(b) \quad f(x) = \frac{1 - \cos x}{x^2} \text{ when } x \neq 0 \text{ and } f(x) = 1 \text{ when } x = 0,$$

at $x=0$.

(ii) $f(c-0)=f(c+0)$, $f(c)$ is not defined.

In such a case, the discontinuity is said to be of the *second kind*.

Exs. (a) $f(x)=(x-c) \sin \frac{1}{x-c}$ at $x=c$.

$$(b) f(x)=\frac{x^2}{x} \text{ at } x=0.$$

Note. In both the above cases, we can make the function continuous by properly defining $f(x)$ at the point in question. Such a discontinuity is, therefore, called a *removable discontinuity*.

II. $f(c-0)$ and $f(c+0)$ are unequal and $f(c)$ is equal to one of them, or to neither or undefined.

Exs. (i) $f(x)=[x-c]$ at $x=c$.

Here $f(c+0)=f(c)=0$; but $f(c-0)=-1$.

(ii) $f(x)=[x-c]-[-(x-c)]$ at $x=c$.

Here $f(c+0)=1$, $f(c-0)=-1$, $f(c)=0$.

(iii) $f(x)=[x-c]+(x-c) \sin \frac{1}{x-c}$ at $x=c$.

Here $f(c+0)=0$, $f(c-0)=-1$, but $f(c)$ is undefined.

Note. A discontinuity of class I or of class II is sometimes called a *simple ordinary discontinuity*.

III. If one or both of $f(c+0)$ and $f(c-0)$ be indefinitely large, then $f(x)$ has an *infinite* discontinuity at $x=c$. $f(c)$ may or may not exist.

Exs. (i) $f(x)=e^{-\frac{1}{x-c}}$, at $x=c$.

(ii) $f(x)=\frac{1}{(x-c)^2}$ at $x=c$.

(iii) $f(x)=\frac{1}{x-c}$ at $x=c$.

6.4. Continuity of composite functions.

Theorem I. The sum, the difference, the product or the quotient of two functions, continuous at a point or in a given interval, forms also a function which is continuous at the point or in the given interval, except in the case of the quotient where the denominator tends to zero.

The theorem follows easily from the corresponding theorem on limits.

Note: A *constant* may be looked upon as a continuous function. A polynomial in x is a continuous function of x . A rational function of x is continuous at all points except where the denominator vanishes.

Theorem II. A continuous function of a continuous function is itself a continuous function.

Let $y = \phi(x)$ be a continuous function of x at any point x_0 of (a, b) and $z = f(y)$ be a continuous function of y at the corresponding point $y_0 = \phi(x_0)$. To prove that $z = f[\phi(x)]$ is a continuous function of x at $x = x_0$. From the continuity of $z = f(y)$ at y_0 , we get $|z - z_0| < \epsilon$, when $|y - y_0| < \delta'$, where $z_0 = f(y_0)$. Again from the continuity of $y = \phi(x)$ at $x = x_0$, we get $|y - y_0| < \delta'$, when $|x - x_0| < \delta$. Combining the two, we have $|z - z_0| < \epsilon$, when $|x - x_0| < \delta$, which proves the theorem.

6'5. Properties of functions continuous in a closed interval.

I. If $f(x)$ be continuous in the *closed* interval (a, b) , it is bounded in the interval.

We can certainly determine an interval (a, X) , extending to the right from a in which $f(x)$ is bounded. For since $f(x)$ is continuous at $x = a$, we can give any positive number ϵ , however small, determine an interval (a, X) throughout which $f(x)$ lies between $f(a) - \epsilon$ and $f(a) + \epsilon$. Obviously $f(x)$ is bounded in this interval.

Now divide the points X of the interval (a, b) into two classes, L and R , putting X in L if $f(x)$ is bounded in (a, X) and in R if this is not the case. It follows from what precedes that L certainly exists; what we propose to prove is that R does not. Suppose that R does exist, and k be the number corresponding to the section whose lower and upper classes are L and R . Since $f(x)$ is continuous at $x = k$, we can, however small the positive number ϵ may be, find an interval $(k - \delta, k + \delta)$ throughout which $f(k) - \epsilon < f(x) < f(k) + \epsilon$. Thus $f(x)$ is bounded in $(k - \delta, k + \delta)$. But $k - \delta$ belongs to L . Therefore $f(x)$ is bounded in $(a, k - \delta)$, and therefore it is bounded in the whole interval $(a, k + \delta)$. Now $k + \delta$ belongs to R and so $f(x)$ is *not* bounded in $(a, k + \delta)$. This contradiction shows that R does not exist. And so $f(x)$ is bounded in the whole interval (a, b) .

II. If $f(x)$ is continuous in the closed interval (a, b) , it attains its upper and lower bounds at least once in the interval. (Weierstrass)

Let M be the upper bound of the function in (a, b) and let us prove the theorem for M . Since M is the upper bound, given any positive ϵ , we can find a value of x for which $f(x) > M - \epsilon$, i.e., $M - f(x) < \epsilon$, or $1/\{M - f(x)\} > 1/\epsilon$. Hence $1/\{M - f(x)\}$ is not bounded and, therefore, not continuous. But $M - f(x)$ is a continuous function and so $1/\{M - f(x)\}$ is continuous at any point at which its denominator does not vanish. Therefore to make $1/\{M - f(x)\}$ discontinuous, there must be at least one point where the denominator vanishes; at this point $f(x) = M$. It may be shown in like manner that there is at least one point where $f(x) = m$, the lower bound.

Note. It is important to observe that the statement need not be true if the function fails to be continuous at the end-points of (a, b) . For example, the function $f(x) = \frac{1}{x}$ has no upper bound in $0 < x \leq 1$, although $f(x)$ is continuous in the interior of this interval. Nor need a discontinuous function assume a greatest or a least value even if it is bounded. As example, consider the very discontinuous function defined by

$$f(x) = x \text{ for irrational } x,$$

$$f(x) = \frac{1}{2} \text{ for rational } x,$$

in the interval $0 \leq x \leq 1$. The function always assumes values between 0 and 1, in fact, as near to 1 and 0 as we please but is never equal to 0 or 1.

Alternative proof.

Let M be the upper bound of $f(x)$ in (a, b) . Divide (a, b) into two equal halves; then M is the upper bound of $f(x)$ in at least one of these halves, say (a_1, b_1) . Divide (a_1, b_1) again into two equal halves; then M is the upper bound in at least one of these halves, say (a_2, b_2) . Proceeding thus we get an infinite sequence of intervals (a, b) , (a_1, b_1) , (a_2, b_2) , ..., (a_n, b_n) , ... in each of which the upper bound is M and each of which is half of the preceding. Now the points $a, a_1, a_2, \dots, a_n, \dots$ and $b, b_1, b_2, \dots, b_n, \dots$ tend to a common limit c for $b_n - a_n = \frac{1}{2^n}(b - a)$ tends to the limit 0 as n tends to ∞ . Since the function is continuous at $x = c$, corresponding to the positive number $\frac{1}{2}\epsilon$ we can find η such that $|f(x) - f(c)| < \frac{1}{2}\epsilon$

for $|x-c| < \eta$. Again by increasing n , we can make $b_n - a_n < 2\eta$, so that $|f(x) - f(c)| < \frac{1}{2}\epsilon$ for values of x in (a_n, b_n) . Now as M is the upper bound of $f(x)$ in (a_n, b_n) , we can make $M - f(x) < \frac{1}{2}\epsilon$ for values of x in (a_n, b_n) . Hence for such values of x , $|M - f(c)| < \epsilon$. Thus M is attained at $x=c$, a point in (a, b) .

The theorem can be proved for the lower bound in exactly the same way.

III. If $f(x)$ is continuous at $x=a$ and $f(a) \neq 0$, then there exists an interval $a-\delta \leq x \leq a+\delta$ about a such that $f(x)$ preserves the same sign within the interval, namely the sign of $f(a)$.

Since $f(a)$ exists, corresponding to any preassigned positive number ϵ , we can determine a positive number δ , such that for every point in the interval $a-\delta \leq x \leq a+\delta$, $|f(x) - f(a)| < \epsilon$, i.e., $f(a) - \epsilon < f(x) < f(a) + \epsilon$. Now taking $\epsilon < |f(a)|$, we notice that $f(a) - \epsilon$ and $f(a) + \epsilon$ are of the same sign as $f(a)$ unless $f(a) = 0$; and consequently $f(x)$ being intermediate between them, maintains the same sign in the interval and the sign is the same as that of $f(a)$.

IV. If $f(x)$ is continuous in the closed interval (a, b) and $f(a)$ and $f(b)$ differ in sign, then $f(x)$ vanishes at least once between a and b . (Bolzano)

To fix ideas suppose $f(a)$ is negative and $f(b)$ is positive. Since $f(x)$ is continuous, it will be negative in the neighbourhood of a and positive in the neighbourhood of b . The set of values of x between a and b which make $f(x)$ positive is bounded below by a and hence possesses an exact lower bound k ; clearly $a < k < b$.

From the definition of lower bound, the values of $f(b)$ must be negative or zero in $a \leq x \leq k$. Since $f(x)$ is continuous at $x=k$,

$\lim_{x \rightarrow k-0} f(x) = f(k)$. Hence $f(k)$ is also negative or zero. We shall show that $f(k)$ cannot be negative. As $f(x)$ is continuous at $x=k$, we can determine an interval $(k-\eta, k+\eta)$ in which $f(x)$ is negative on the supposition that $f(k)$ is negative (by III above), $f(x)$ being negative for values of x which lie between k and $k+\eta$, k cannot be the lower bound of the set of values of x between a and b which make $f(x)$ positive. From this contradiction it follows that $f(k) = 0$, and the theorem is proved.

V. If $f(a)$ and $f(b)$ are unequal and $f(x)$ is continuous in the closed interval (a, b) , $f(x)$ assumes at least once every value between $f(a)$ and $f(b)$.

Let λ be any number between $f(a)$ and $f(b)$. The continuous function $F(x) = f(x) - \lambda$ has opposite signs for $x = a$ and $x = b$. Hence by IV above, $F(x)$ vanishes at least once between a and b , so that for this value of x , $f(x) = \lambda$.

VI. Let $f(x)$ be continuous in the closed interval (a, b) and ϵ any positive number, however small. Then the interval (a, b) can always be broken up into a finite number of partial intervals in such a way that for any two values of x whatever, x' and x'' which belong to the same partial interval, we always have $|f(x') - f(x'')| < \epsilon$. (Heine)

Suppose this were not true, that is, it is not possible to divide (a, b) into a finite number of partial intervals satisfying the above statement. Let $c = \frac{1}{2}(a + b)$; at least one of the intervals (a, c) , (c, b) would have the same property as (a, b) i.e., it would be impossible to break it up into a finite number of partial intervals satisfying the statement of the theorem. Let the partial interval in which the theorem is not true be called (a_1, b_1) . Substituting it for (a, b) and using the process repeatedly, we form an infinite sequence of intervals (a, b) , (a_1, b_1) , (a_2, b_2) , ... (a_n, b_n) , ... , each of which is half of the preceding and in each of which the theorem is false.

According to our supposition then, we can always find in the interval (a_n, b_n) , two numbers, x' and x'' , such that

$$|f(x') - f(x'')| > \epsilon, \quad \dots \dots (\alpha)$$

Now $a \leq a_1 \leq a_2 \leq \dots \leq a_n \leq \dots \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1 \leq b$, and $b_n - a_n = \frac{1}{2^n}(b - a)$. As the sequence $\{a_n\}$ is monotone increasing and bounded above and the sequence $\{b_n\}$ is monotone decreasing and bounded below, they are both convergent. Further $b_n - a_n \rightarrow 0$ as $n \rightarrow \infty$ and therefore $\{a_n\}$ and $\{b_n\}$ tend to a common limit λ such that $a \leq \lambda \leq b$.

Let λ be an interior point of (a, b) .

Since $f(x)$ is continuous at $x=\lambda$, we can find a positive number δ such that $|f(x)-f(\lambda)| < \frac{1}{2}\epsilon$ when $|x-\lambda| \leq \delta$.

x' and x'' being any two numbers of $(\lambda-\delta, \lambda+\delta)$, we have

$$|f(x')-f(\lambda)| < \frac{1}{2}\epsilon, \quad |f(x'')-f(\lambda)| < \frac{1}{2}\epsilon,$$

and so

$$\begin{aligned} |f(x')-f(x'')| &= |f(x')-f(\lambda)+f(\lambda)-f(x'')| \\ &\leq |f(x')-f(\lambda)| + |f(x'')-f(\lambda)| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

Thus the theorem is true in $(\lambda-\delta, \lambda+\delta)$.

Let the integer n be so chosen that (a_n, b_n) lies within $(\lambda-\delta, \lambda+\delta)$. The theorem being true in $(\lambda-\delta, \lambda+\delta)$, it is also true in (a_n, b_n) of which it is a part and this contradicts the supposition (α). Hence the assumption that the theorem is false is not true. Thus the theorem is proved to be true.

Let λ coincide with a .

There is a positive number δ such that

$$|f(x)-f(a)| < \frac{1}{2}\epsilon \text{ when } a \leq x \leq a+\delta.$$

If x' and x'' be any two numbers of $(a, a+\delta)$, it follows as above that $|f(x')-f(x'')| < \epsilon$; so the theorem is true for $(a, a+\delta)$. If now n be so chosen that $b_n < a+\delta$, then (a_n, b_n) lies within $(a, a+\delta)$ and so the theorem is true in (a_n, b_n) which is a part of $(a, a+\delta)$. Here again we have a contradiction, proving the theorem to be true.

The case where λ coincides with b may be similarly dealt with.

Corollary I.

Let $\frac{\epsilon}{2}$ be any arbitrary positive number. We can break up (a, b) into a finite number of partial intervals such that x' and x'' being any two values of x in the same partial interval, we have

$$|f(x')-f(x'')| < \frac{1}{2}\epsilon.$$

Let δ be the least of these partial intervals and consider two values, x' and x'' , of x in (a, b) such that $|x'-x''| \leq \delta$. Then x' and x'' may be in the same interval, in which case $|f(x')-f(x'')| < \frac{1}{2}\epsilon < \epsilon$.

If x' and x'' do not lie in the same interval, they must lie in consecutive intervals as $|x'-x''| \leq \delta$. If now c be the point which

separates the two intervals, we have $|f(x') - f(c)| < \frac{1}{2}\epsilon$,
 $|f(x'') - f(c)| < \frac{1}{2}\epsilon$ and therefore $|f(x') - f(x'')| < \epsilon$.

Thus given ϵ , we can find δ , depending on ϵ only, such that if
 $|x' - x''| \leq \delta$, then $|f(x') - f(x'')| < \epsilon$.

Corollary II.

Given ϵ , we can divide the closed interval (a, b) into a finite number of sub-intervals such that x' and x'' being any two points in the same sub-interval

$$|f(x') - f(x'')| < \epsilon.$$

Let δ be the least of the sub-intervals in which (a, b) is divided and let η_r be any arbitrary sub-interval of (a, b) such that $\eta_r < \delta$. If x' and x'' be two points in η_r , we have

$$|x' - x''| < \delta \text{ and } |f(x') - f(x'')| < \epsilon.$$

Let M_r and m_r be the upper and lower bounds of $f(x)$ in η_r , and as $f(x)$ is continuous in η_r , these bounds are attained in η_r . Let x' and x'' be the two points in η_r , where these bounds are attained, i.e., let $M_r = f(x')$ and $m_r = f(x'')$. Then,

$M_r - m_r = f(x') - f(x'') = |f(x') - f(x'')| < \epsilon$ when $|x' - x''| < \delta$.
 Thus we get the theorem :

If a function is continuous in a closed interval (a, b) and ϵ any given positive number, there exists a positive number δ such that the oscillation of $f(x)$ in every sub-interval of length less than δ is less than ϵ .

6.6. Uniform Continuity.

Let $f(x)$ be continuous in the closed interval (a, b) and x' a point of this interval. If for a given arbitrary positive ϵ , we can find a positive number δ such that

$$|f(x) - f(x')| < \epsilon \text{ when } |x - x'| < \delta, \quad \dots \quad (1)$$

$f(x)$ is continuous at the point x' . Here δ depends upon both ϵ and x' . Putting x'' for x in (1), we may write it as

$$|f(x'') - f(x')| < \epsilon \text{ when } |x'' - x'| < \delta. \quad \dots \quad (2)$$

If for a given ϵ , we can find δ , depending on ϵ only, such that
 $|f(x') - f(x'')| < \epsilon$ when $|x' - x''| < \delta$,
 the same δ serving for all points x' of (a, b) for the given ϵ , $f(x)$ is said to be *uniformly continuous* in (a, b) . The corollary I to Heine's

theorem (Art. 6'5) shows that given ϵ we can find δ depending on ϵ only such that

$$|f(x') - f(x'')| < \epsilon \text{ if } |x' - x''| < \delta.$$

Hence we have the following result :

A function continuous in a closed interval is uniformly continuous in the interval.

6'7. Inverse functions.

Given $y = f(x)$, it is sometimes possible to solve this equation and express x as function of y , say $x = \phi(y)$.

Then $\phi(y)$ is called the *inverse* of $f(x)$.

However given $f(x)$, there may arise difficulties and complications in the process of determining the inverse function. For example :

(a) The equation $y = f(x)$ may not be solvable for x in terms of y .

(b) The equation $y = f(x)$ may not always give a unique value of x in terms of y .

In view of these difficulties, the following simple test will enable us to find if a given function admits of an inverse or not.

If a function is continuous and strictly monotonic in a given interval, it admits of an inverse.

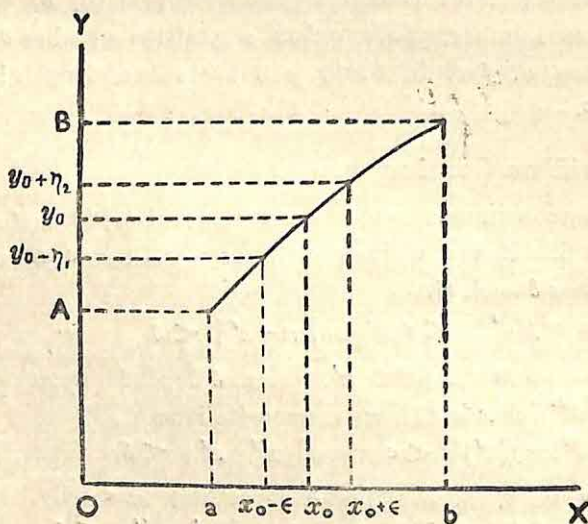


Fig. 14

Let the function $f(x)$ defined in the interval (a, b) be continuous

and monotonic in the strict sense in (a, b) . For instance, let $y=f(x)$ be continuous and continually increase from A to B . Then to every value of y in (A, B) , there corresponds one, and only one, value of x in (a, b) .

This value of x is a function of y , say $\phi(y)$, which is itself continually increasing in the interval (A, B) . The function $\phi(y)$ is the *inverse* of the function $f(x)$. We shall now show that $\phi(y)$ is a *continuous* function of y in the interval in which it is defined.

For let y_0 be any number between A and B and x_0 the corresponding value of x . Also let ϵ be an arbitrary positive number such that $x_0 - \epsilon$ and $x_0 + \epsilon$ lie in (a, b) . (See diagram on page 84).

Let $y_0 - \eta_1$ and $y_0 + \eta_2$ be the values of y corresponding respectively to the values $x_0 - \epsilon$ and $x_0 + \epsilon$ of x . Then if η be a positive number less than the smaller of η_1 and η_2 , it is clear that

$$\begin{aligned} |x - x_0| &< \epsilon \text{ when } |y - y_0| \leq \eta. \\ \therefore |\phi(y) - \phi(y_0)| &< \epsilon \text{ when } |y - y_0| \leq \eta. \end{aligned}$$

Thus $\phi(y)$ is continuous at $y = y_0$.

6.8. Other definitions of continuity at a point.

Heine's definition : Let $\{x_n\}$ be a convergent sequence of points in (a, b) converging to the limit c . Thus function $f(x)$ defined in (a, b) is said to be continuous at $x=c$, if $\{f(x_n)\}$ converges to $f(c)$.

Another definition : Let M and m be the upper and lower bounds of $f(x)$ in the interval $(c - \delta_1, c + \delta_2)$ within (a, b) . Then $M - m$ is the oscillation of $f(x)$ in $(c - \delta_1, c + \delta_2)$. If δ_1 and δ_2 tend to zero simultaneously and independently, the limit of $M - m$ is called the oscillation of $f(x)$ at $x=c$.

A function is said to be continuous at a point if its oscillation at the point is zero.

The three definitions can be proved to be equivalent.

6.9. Infinitesimals.

An infinitesimal is a variable whose limit is zero.

A fixed number however small is not an infinitesimal.

A variable x is said to be infinitesimal with respect to another variable y , when the ratio $\frac{x}{y}$ tends to zero. This is symbolically represented by $x=o(y)$.

Two infinitesimals, x and y are said to be of the *same order* if the ratio $\frac{x}{y}$ tends to a finite non-zero limit. In symbols, this is expressed as $x = O(y)$.

Two infinitesimals x and y are *equivalent* if $\frac{x}{y} \rightarrow 1$.

A particular infinitesimal x is chosen and called the *principal infinitesimal*. By means of it all other infinitesimals are classified. Thus an infinitesimal y of the same order as x is said to be of the *first order* and if y be of the same order as x^r , i.e., if $\text{Lt } \frac{y}{x^r} = l (\neq 0)$ then y is an infinitesimal of order r .

If a function be expressible as the sum of a number of infinitesimals of different orders, the one of *lowest* order is called the *principal part*.

We have the following theorems on infinitesimals :

Theorem I. The limit of the quotient of two infinitesimals is unaltered when we substitute for them their equivalent infinitesimals.

Let the infinitesimals x and y be respectively equivalent to x' and y' , so that $\frac{x}{x'} \rightarrow 1$ and $\frac{y}{y'} \rightarrow 1$.

$$\text{Now } \frac{x'}{y'} = \frac{x}{y} \cdot \frac{x'}{x} \cdot \frac{y}{y'}; \quad \therefore \text{Lt } \frac{x'}{y'} = \text{Lt } \frac{x}{y} \cdot \text{Lt } \frac{x'}{x} \cdot \text{Lt } \frac{y}{y'} = \text{Lt } \frac{x}{y}.$$

Theorem II. The difference of two equivalent infinitesimals is an infinitesimal with respect to each of them.

Let x and x' be equivalent infinitesimals and δ their difference. Then $\delta = x - x'$ may be written as $\frac{\delta}{x'} = \frac{x}{x'} - 1 \rightarrow 0$. Hence δ is an infinitesimal with respect to x' . Conversely since $\frac{x}{x'} = 1 + \frac{\delta}{x'}$ & $\text{Lt } \frac{\delta}{x'} = 0$, it follows that $\frac{x}{x'}$ tends to 1.

Theorem III. The limit of the sum of an infinite number of infinitesimals of the same sign is unaffected if the infinitesimals are replaced by equivalent ones.

Suppose the infinitesimals $x_1, x_2, \dots$ are all positive. Given any positive number ϵ in advance, we must have

$$1 - \epsilon < \frac{y_n}{x_n} < 1 + \epsilon.$$

And since x_n is positive,

$$(1 - \epsilon) \sum_1^n x_n < \sum_1^n y_n < (1 + \epsilon) \sum_1^n x_n.$$

Since $\sum x_n$ is clearly not zero, being the sum of positive terms, we have

$$1 - \epsilon < \frac{\sum_1^n y_n}{\sum_1^n x_n} < 1 + \epsilon.$$

Hence we conclude

$$\text{Lt}_{n \rightarrow \infty} \frac{y_1 + y_2 + \dots + y_n}{x_1 + x_2 + \dots + x_n} = 1,$$

or, in other words, $\sum_1^n x_n$ and $\sum_1^n y_n$ have the same limit, finite, zero or infinite.

From the above theorem it follows easily that if in any sum or quotient involving infinitesimals we replace each term by its principal part, limits are not affected in any way.

Examples :

1. What is the nature of the discontinuities at $x=0$ of the functions : $\frac{\sin x}{x}$, $[x] + [-x]$, $\text{cosec } x$, $\text{cosec } \frac{1}{x}$, $\frac{\sin \frac{1}{x}}{\sin \frac{1}{x}}$?

2. Show that $f(x) = \sin \frac{1}{x}$ is discontinuous at $x=0$, no matter how $f(0)$ is defined.

3. Show that a polynomial in x is a continuous function of x .

4. Determine whether the following functions are continuous at $x=0$:

(a) $f(x) = (x^4 + 2x^3 + x^2) / \sin x$, $f(0) = 0$.

(b) $f(x) = (x^4 + 2x^3 + 3x) / \sin x$, $f(0) = 0$.

5. If $\phi(x)=0$ when $x=0$,

$$\phi(x)=\frac{1}{2}-x \text{ when } 0 < x < \frac{1}{2},$$

$$\phi(x)=\frac{1}{2} \text{ when } x=\frac{1}{2},$$

$$\phi(x)=\frac{3}{2}-x \text{ when } \frac{1}{2} < x < 1,$$

$$\phi(x)=1 \text{ when } x=1,$$

show that $\phi(x)$ assumes every value between 0 and 1 once and only once as x increases from 0 to 1. Find the discontinuities of $\phi(x)$.

(C. H. 1965)

6. Examine $f(x)$ for continuity at $x=0$ where

$$f(x) = \frac{1}{1 - e^{\frac{1}{x}}} \text{ when } x \neq 0,$$

$$\text{and } f(x)=0 \text{ when } x=0.$$

(C. H. 1965)

7. Examine if the function $f(x)$ defined as follows is continuous at $x=1$:

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ x-2, & 1 < x \leq 2. \end{cases} \quad (\text{C. H. 1963})$$

8. Prove that if $f(x)$ is continuous at $x=a$ and for every positive δ , there is a point in $|x-a| < \delta$ at which $f(x)=0$, then $f(a)=0$.

(C. H. 1963)

9. If $f(x)$ is continuous in the closed interval (a, b) , prove that

$$\frac{\lambda f(a) + \mu f(b)}{\lambda + \mu} = f(c),$$

where c is some value of x in (a, b) , λ/μ being positive.

10. The function $f(x)=x$, when x is rational and $f(x)=0$, when x is irrational is continuous at $x=0$ but is discontinuous for all other values of x .

11. A function $f(x)$ is defined for positive values of x as follows :

$$f(x)=0, \text{ when } x \text{ is irrational,}$$

$$f(x)=\frac{1}{q}, \text{ when } x \text{ is rational of the form } \frac{p}{q} \text{ in its lowest terms ;}$$

show that $f(x)$ is continuous for irrational values of x and discontinuous for rational values of x .

[Let $x = \frac{p}{q}$ so that $f(x) = \frac{1}{q}$. In the interval $\left(\frac{p}{q} - \delta, \frac{p}{q} + \delta\right)$, there is an infinite number of irrational points at each of which $f(x) = 0$, we have

$$\left| f(x) - f\left(\frac{p}{q}\right) \right| = \left| 0 - \frac{1}{q} \right| = \frac{1}{q}$$

and this cannot be made less than any arbitrary ϵ . Hence $f(x)$ is discontinuous at all rational points.

Let c be an irrational point, so that $f(c) = 0$. Enclosing c in an interval $(c - \delta', c + \delta')$, there is an infinite number of rational points x in it. Since x is variable, $f(x) = \frac{1}{q}$ is also variable, i.e., q is variable. Giving q the values 1, 2, 3, ..., we can find a value $q = m$ such that for $q \geq m$, $\frac{1}{q} < \epsilon$ and for $q = 1, 2, 3, \dots, m-1$, $\frac{1}{q} > \epsilon$. The number of values of q which makes $\frac{1}{q} > \epsilon$ is finite and the number which makes $\frac{1}{q} < \epsilon$ is infinite. Also the points for which $\frac{1}{q} > \epsilon$ are farther off from those for which $\frac{1}{q} < \epsilon$. It then we cut off from $(c - \delta', c + \delta')$ a sub-interval $(c - \delta, c + \delta)$ so as to exclude the finite number of points for which $\frac{1}{q} > \epsilon$, we have in $(c - \delta, c + \delta)$ an infinite number of rational points where $\frac{1}{q} < \epsilon$.

$$\therefore |f(x) - f(c)| = \left| \frac{1}{q} - 0 \right| = \frac{1}{q} < \epsilon. \quad \dots (1)$$

Again if x be an irrational point in $(c - \delta, c + \delta)$, we have

$$|f(x) - f(c)| = |0 - 0| = 0 < \epsilon. \quad \dots (2)$$

Hence, from (1) and (2), we find that $f(x)$ is continuous at $x = c$, that is, at all irrational points.]

12. $f(x)$ is continuous in $(-1, 1)$ and assumes rational values only; if $f(0) = 0$, prove that $f(x) = 0$ everywhere in $(-1, 1)$.

13. $f(x)$ is continuous in $(0, 1)$ and assumes rational values only in the interval. If $f(x) = \frac{1}{2}$, when $x = \frac{1}{2}$, show that $f(x) = \frac{1}{2}$ everywhere in $(0, 1)$.

14. Discuss the continuity of
- $f(x)$
- where

 $f(x)=0$, when x is irrational or zero, $=\frac{1}{q^3}$, when $x=\frac{p}{q}$ in its lowest terms.

15. Show that the function defined by

 $f(x)=x$, when x is rational, $=1-x$, when x is irrational

assumes every value between 0 and 1 once and only once as x increases from 0 to 1, but is discontinuous for every value of x except $x=\frac{1}{2}$.

(C. H. 1965)

16. If
- $f(x)=x^2+2x+5$
- in
- $(-1, 1)$
- , find
- δ
- so that

$$|f(x_1)-f(x_2)| < \epsilon \text{ whenever } |x_1-x_2| < \delta.$$

$$[|f(x_1)-f(x_2)| = |x_1-x_2| |x_1+x_2+2| \\ \leq |x_1-x_2| \{|x_1| + |x_2| + 2\}$$

$$< 4|x_1-x_2| < \epsilon, \text{ if } |x_1-x_2| < \frac{\epsilon}{4}.$$

$$\text{Hence } \delta = \frac{\epsilon}{4}.$$

17. If $f(x)$ is continuous in the interval $a \leq x \leq b$, prove that it is bounded in the interval. Give an example to show that if $f(x)$ is continuous in $a < x < b$, then it is not necessarily bounded.

(C. H. 1962)

18. If $f(x)$ be continuous and strictly increasing in $a \leq x \leq b$, prove that there exists one and only one value of x which satisfies the equation $f(x)=y$ where y is any number between $f(a)$ and $f(b)$, and that the inverse function so defined is continuous in the corresponding interval.

(C. H. 1962)

19. If $f(x)$ be a continuous function of x and satisfies the relation $f(x+y)=f(x)+f(y)$, show that $f(x)=ax$ where a is a constant.

$$[x=y=0 \text{ gives } f(0)=0 \quad \dots \quad \dots \quad \dots \quad (1)$$

$$y=-x \text{ gives } f(-x)=-f(x). \quad \dots \quad \dots \quad (2)$$

If x be a positive integer, $f(x)=f(1+1+\dots$ to x terms)

$$=f(1)+f(1)+\dots \text{ to } x \text{ terms} = xf(1)=ax, \text{ where } f(1)=a.$$

If x be a negative integer, writing $x=-y$, so that y is a positive integer, $f(x)=f(-y)=-f(y)=-ay=ax$.

Let x be a rational number $\frac{p}{q}$, where q is positive. Then

$$f(p) = f\left(q\frac{p}{q}\right) = f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \dots \text{to } q \text{ terms} = qf\left(\frac{p}{q}\right),$$

or, $ap = qf\left(\frac{p}{q}\right)$; $\therefore f(x) = f\left(\frac{p}{q}\right) = a\frac{p}{q} = ax$, where x is any rational number.

Next let x be any real number and let $\{x_n\}$ be a sequence of rational numbers such that $\text{Lt } x_n = x$. x_n being rational, $f(x_n) = ax_n$, $f(x)$ being a continuous function, as n tends to ∞ , we have $f(x) = ax$, for all real values of x .]

20. If $f(x) = \text{Lt}_{n \rightarrow \infty} \frac{\log(2+x) - x^{2n} \sin x}{1+x^{2n}}$, explain why the function does not vanish anywhere in $\left(0, \frac{\pi}{2}\right)$ although $f(0)$ and $f\left(\frac{\pi}{2}\right)$ differ in sign. (Bombay Univ. 1934)

[Hints : For $x > 1$, $f(x) = -\sin x$; ... (1)

for $-1 < x < 1$, $f(x) = \log(2+x)$; ... (2)

for $x=1$, $f(x) = f(1) = \frac{\log 3 - \sin 1}{2}$.

So $\text{Lt}_{x \rightarrow (1+0)} f(x) = -\sin 1$ and $\text{Lt}_{x \rightarrow (1-0)} f(x) = \log 3$. $\therefore \text{Lt}_{x \rightarrow 1} f(x)$ does not exist, i.e., $f(x)$ is discontinuous at $x=1$ (3)

Since $\frac{\pi}{2} > 1$, from (1) we have $f\left(\frac{\pi}{2}\right) = -1 < 0$;

also since $-1 < 0 < 1$, we have from (2), $f(0) = \log 2 > 0$.

Now in $\left(0, \frac{\pi}{2}\right)$, $f(x)$ may be defined as follows :

$$\begin{aligned} f(x) &= \log(2+x) \text{ when } 0 \leq x < 1, \\ &= \frac{1}{2}(\log 3 - \sin 1) \text{ when } x=1, \\ &= -\sin x \text{ when } 1 < x \leq \frac{\pi}{2}, \end{aligned}$$

so that $f(x)$ does not vanish anywhere in $0 \leq x \leq \frac{\pi}{2}$.

The reason is that $f(x)$ is not continuous in the closed interval $\left[0, \frac{\pi}{2}\right]$.

CHAPTER 7

DERIVATIVES AND DIFFERENTIALS

7.1. Derivative.

Let $y=f(x)$ be a finite, single-valued function defined in the interval (a, b) and x a point of this interval. Let us give to x an increment $\Delta x=h$, positive or negative. Then the corresponding increment Δy of the function is $f(x+h)-f(x)$ and the ratio of the two increments is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h}.$$

If this ratio tends to a limit, finite or infinite, as h tends to zero in any manner whatsoever, this limit is called the *derivative* or the *derived function* or the *differential coefficient* of $f(x)$ at the point x and is denoted by $f'(x)$ (Lagrange), by $Df(x)$ (Arbogast) or by $D_x f(x)$ (Cauchy). Thus

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}.$$

If the ratio $\frac{\Delta y}{\Delta x}$ tends to a limit when h tends to zero by positive values only, the limit is called the *derivative to the right* or the *right-hand derivative* of the function $f(x)$ at x and is usually denoted by $Rf'(x)$. Similarly the *derivative to the left* or the *left-hand derivative* is the limit of the ratio when h tends to zero by negative values only and is denoted by $Lf'(x)$. When these derivatives are equal, the function is said to have a unique derivative or simply a derivative at the point x . If, however, the two derivatives are unequal, we say that the derivative of $f(x)$ at the point x does not exist.

If $f(x)$ has a unique derivative at all points of the open interval (a, b) and further has a derivative to the right of a and a derivative to the left of b , it is said to be *derivable* in the closed interval (a, b) . If the derivative is always finite, $f(x)$ is *finitely derivable* in (a, b) .

7.2. Differential.

We shall say that a function $f(x)$ is *differentiable* at the point x , if the function is finite and determinate in the neighbourhood of the point and if on giving x an arbitrary increment Δx , the corresponding increment $\Delta f(x)$ of $f(x)$ can be decomposed into the sum of two terms :

$$\Delta f(x) = A \Delta x + \epsilon \Delta x, \dots \dots \dots (1)$$

A being independent of Δx and ϵ tending to zero with Δx . Then the first term, which is simply proportional to Δx , is called the *differential* of y and is represented by dy or $df(x)$. (Leibnitz)

We have therefore

$$\begin{aligned} df(x) &= A \Delta x, \\ \Delta f(x) &= df(x) + \epsilon \Delta x. \end{aligned}$$

When Δx tends to zero, we get from (1), $\text{Lt } \frac{\Delta f(x)}{\Delta x} = A$, which proves that if $f(x)$ is differentiable, $f'(x)$ exists and has a finite and determinate value A . Conversely, if $f'(x)$ has a finite value, we get equation (1) from the definition of the derivative. Therefore *the necessary and sufficient condition that the function $f(x)$ is differentiable at the point x is that it has a finite and determinate derivative at this point.* We have, in this case,

$$df(x) = f'(x) \cdot \Delta x. \dots \dots (2)$$

The differential of a function is the product of the derivative, supposed existent and finite, by an arbitrary increment Δx attributed to the independent variable x .

In the particular case where $f(x)$ reduces to x and therefore $f'(x) = 1$, the equation (2) reduces to

$$dx = \Delta x.$$

Thus the differential of the independent variable is the same as the arbitrary increment attributed to this variable. The formula (2) can thus be replaced in the general case by

$$df(x) = f'(x) \cdot dx \dots \dots (3)$$

Hence the differential of a function is the product of its derivative by the differential of the independent variable. On dividing by dx , we find

$$f'(x) = \frac{df(x)}{dx} = \frac{dy}{dx}.$$

The derivative of a function is thus the ratio of the differential of the function to the differential of the independent variable. This gives a new expression for the derivative due to Leibnitz and one that is frequently used.

Note. The substitution of dx for Δx in equation (2) is not absolutely necessary but it is customary to do so; and this in a sense is justified. Equation (3) is more general than (2) in the sense that in (2) x is the independent variable, but equation (3) is not subject to this restriction.

7.3. Geometrical representation of the derivative and the differential.

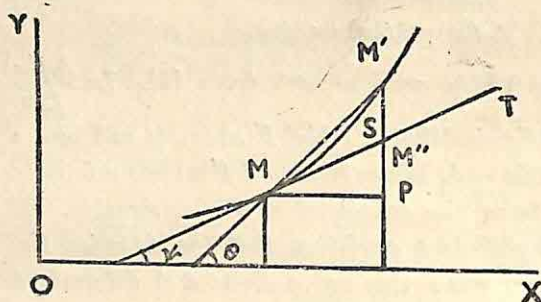


Fig. 15

The problem of drawing tangents to plane curves leads us to the consideration of ratios of infinitesimals and to the definition of the derivative. In fact, we proceed to show that the determination of the tangent to a plane curve is the same as finding the derivative of a function.

Let $y=f(x)$ be the equation of a plane curve MSM' referred to rectangular axes OX and OY . Let M be the point (x, y) and consider the chord MM' drawn through M . When the point M' moves on the curve and approaches indefinitely near to M , the chord MM' tends to a limiting position MT which is the tangent to the curve at the point M .

The point M being given, the tangent MT is known when we know the angle ψ which it makes with the axis of x . Let θ be the angle which the chord MM' makes with the x -axis, and let the

co-ordinates of M' be $(x + \Delta x, y + \Delta y)$. Draw MP parallel to OX ; we have

$$\tan \theta = \frac{M'P}{MP} = \frac{\Delta y}{\Delta x}.$$

When M' approaches indefinitely near to M , θ tends to ψ ; and therefore

$$\tan \psi = \text{Lt} \frac{\Delta y}{\Delta x} = \text{Lt}_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

The derivative of the function $f(x)$ at the point x is thus the trigonometrical tangent of the angle which the tangent to the curve at the point (x, y) makes with the axis of x .

The differential of a function is also capable of geometrical representation as in the case of the derivative. Consider the curve $y = f(x)$. Draw the tangent MT to the curve at M . Let Δx be an arbitrary increment of x and let M'' be the point on this tangent whose abscissa is $x + \Delta x$. We have $\Delta x = MP$ and the corresponding increment of the ordinate of this tangent is PM'' . Now $\tan \psi = \frac{PM''}{PM}$.

But we have seen that $\tan \psi = f'(x)$. We thus get

$$PM'' = f'(x) \cdot PM = f'(x) \cdot \Delta x = df(x).$$

Thus the differential of $f(x)$ is equal to the increment of the ordinate of the tangent to the curve $y = f(x)$, when we pass from the abscissa x of the point of contact to another abscissa $x + \Delta x$, where Δx is an infinitesimal and therefore tends to zero.

7.4. Theorem. A function which has a *finite* derivative for a given value of x is continuous at that point.

Let ϵ be a positive quantity which tends to 0 with h ;

we have
$$\frac{f(x+h) - f(x)}{h} = f'(x) + \epsilon;$$

$$\therefore \text{Lt}_{h \rightarrow 0} [f(x+h) - f(x)] = \text{Lt}_{h \rightarrow 0} [h\{f'(x) + \epsilon\}] = 0,$$

as $f'(x)$ is finite. Hence $f(x)$ is continuous at the point.

It follows that if $f(x)$ is finitely derivable in a given interval, it is continuous in the interval.

The converse of this proposition is not necessarily true. We cannot say that a continuous function has a derivative for every

value of x for which it is continuous. We illustrate this by means of the following examples :

(a) Let us consider the curve formed by two straight lines PA and PB meeting at an angle.

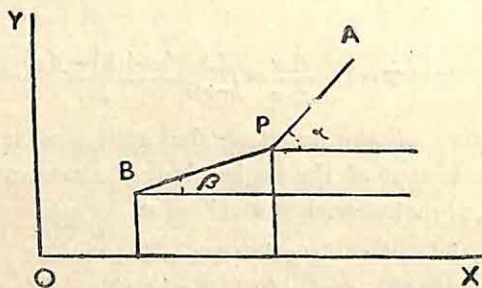


Fig. 16

It is easy to see that at the point P of this curve

$$\frac{f(x+h) - f(x)}{h}$$

tends to the limit $\tan \alpha$ as h tends to 0 through positive values and to the limit $\tan \beta$ as h tends to 0 through negative values. Thus although the curve is continuous at P , it has no unique derivative there.

(b) Consider the function $f(x) = |x|$. It is continuous at $x=0$, but has no derivative there.

For continuity $|f(x) - f(0)| = |x| < \epsilon$ whenever $|x - 0| < \delta$ i.e., $|x| < \delta$. Choosing $\delta = \epsilon$ we see that $f(x)$ is continuous at $x=0$.

For derivability at $x=0$.

$$\begin{aligned} \frac{f(0+h) - f(0)}{h} &= \frac{|h|}{h} = 1 \text{ when } h > 0 \\ &= -1 \text{ when } h < 0. \end{aligned}$$

Thus $Rf'(0) = 1$ and $Lf'(0) = -1$; and hence $f'(0)$ does not exist.

(c) Let $f(x) = x \sin \frac{1}{x}$ when $x \neq 0$, $f(0) = 0$.

Here $\text{Lt } f(0+h) = \text{Lt } \left[h \sin \frac{1}{h} \right] = 0 = f(0)$; so $f(x)$ is continuous

at $x=0$. But $f'(0) = \text{Lt} \sin \left(\frac{1}{h} \right)$ and $\sin \frac{1}{h}$ does not tend to any limit as h tends to 0. Therefore $f'(0)$ does not exist.

Note. The discoverers of the infinitesimal calculus confidently believed that a continuous function had a derivative *in general*. Attempts at proof were even made; but these were of course fallacious. Finally Weierstrass succeeded in settling the question conclusively by giving examples of continuous functions which do not possess derivatives for any values of the variable whatever.

7.5. On the sign of the derivative: Increasing and decreasing functions.

Let $f(x)$ be differentiable in (a, b) and c an interior point of (a, b) . Then

$$f'(c) = \text{Lt}_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}.$$

Corresponding to any positive ϵ , there is a positive number δ such that

$$\left| \frac{f(c+h) - f(c)}{h} - f'(c) \right| < \epsilon \text{ when } 0 < |h| \leq \delta,$$

$$\text{i.e., } f'(c) - \epsilon < \frac{f(c+h) - f(c)}{h} < f'(c) + \epsilon.$$

Let $f'(c) > 0$; choose $\epsilon < f'(c)$; then $f'(c) - \epsilon$ and $f'(c) + \epsilon$ are both positive; therefore

$$\frac{f(c+h) - f(c)}{h} > 0 \text{ when } 0 < |h| \leq \delta,$$

$$\begin{aligned} \text{i.e., } f(c+h) &> f(c) \text{ when } 0 < h \leq \delta, \\ \text{and } f(c+h) &< f(c) \text{ when } -\delta \leq h < 0. \end{aligned}$$

Putting $c+h=x$ so that $c-\delta \leq x \leq c+\delta$, we see that $f(x) > f(c)$ if $x > c$ and $f(x) < f(c)$, if $x < c$.

Thus if $f'(c) > 0$, we can find a neighbourhood of c such that $f(x_2) > f(x_1)$ when $c-\delta < x_1 < x_2 < c+\delta$.

The function $f(x)$ is then called an *increasing function* in $(c-\delta, c+\delta)$.

Next let $f'(c) < 0$, choose $\epsilon < |f'(c)|$; then $f'(c) - \epsilon$ and $f'(c) + \epsilon$

are both < 0 , $\therefore \frac{f(c+h)-f(c)}{h} < 0$ so that $f(c+h)-f(c)$ and h are of opposite signs. Now if $c+h=x$, so that $c-\delta \leq x \leq c+\delta$, we see that $f(x) < f(c)$, if $x > c$ and $f(x) > f(c)$ if $x < c$. Thus if $f'(c) < 0$, we can find a neighbourhood of c such that $f(x_2) < f(x_1)$ when $c-\delta < x_1 < x_2 < c+\delta$.

The function $f(x)$ is then called a *decreasing function* in $(c-\delta, c+\delta)$.

For the end-points a and b , we have only to restrict the intervals about them.

Combining the two results we have the theorem :

According as $f'(c)$ is positive or negative, $f(x)$ is increasing or decreasing in a suitably restricted neighbourhood of c .

Cor. 1. If $f'(x) > 0$ for all values of x in a certain interval, then $f(x)$ is a *strictly* increasing function throughout the interval.

Cor. 2. If $f'(x) > 0$ throughout the interval (a, b) and $f(a) \geq 0$, then $f(x)$ is positive throughout the interval.

7'6. Rolle's Theorem.

If $f(x)$ be (i) continuous in the closed interval $a \leq x \leq b$ (ii) derivable in the open interval $a < x < b$, further (iii) if $f(a)=f(b)=0$, then there is a value ξ of x between a and b , i.e., $a < \xi < b$, such that $f'(\xi)=0$.

$f(x)$ vanishes, by hypothesis, for $x=a$ and $x=b$. If it vanishes at every point of the interval (a, b) , its derivative also vanishes at every point of the interval and the theorem is obvious.

If not, $f(x)$ being continuous in $a \leq x \leq b$, is bounded in (a, b) and attains its bounds in (a, b) . Suppose the upper bound is attained at ξ ($a < \xi < b$). Then we have, if h be positive

$$f(\xi \pm h) \leq f(\xi).$$

$\therefore f(\xi+h)-f(\xi) \leq 0$. Hence $\{f(\xi+h)-f(\xi)\}/h \leq 0$. Making h tend to 0, we get $f'(\xi) \leq 0$ (1)

Again since $\{f(\xi-h)-f(\xi)\}/(-h) \geq 0$, making h tend to 0, we have $f'(\xi) \geq 0$ (2)

i.e., $R f'(\xi) \leq 0$ and $L f'(\xi) \geq 0$.

But $f'(\xi)$ exists by hypothesis. This is possible only if $Rf'(\xi) = Lf'(\xi) = 0$. Thus $f'(\xi) = 0$ where $a < \xi < b$.

If the lower bound is attained at ξ , we would similarly have $f'(\xi) = 0$.

Alternative form of Rolle's theorem.

If $f(x)$ be continuous in $a \leq x \leq b$, derivable in $a < x < b$ and $f(a) = f(b)$, then $f'(\xi) = 0$, where $a < \xi < b$.

This can be established directly as above. It can also be deduced as a corollary to the above.

For let $f(a) = f(b) = A$. Consider the function $F(x) = f(x) - A$. $F(x)$ satisfies all the conditions of the theorem; therefore $F'(\xi) = 0$ for $x = \xi$. But $F'(x) = f'(x)$. $\therefore f'(\xi) = 0$, where $a < \xi < b$.

Note. Rolle enunciated the theorem in the case of a polynomial only; the above is a generalised form of the theorem.

7.7. Theorem of Darboux.

If $f(x)$ be finitely derivable in (a, b) , $f'(x)$ cannot pass from one value to another without passing through all intermediate values.

We first consider the case where $f'(a)$ and $f'(b)$ are of opposite signs and show that $f'(x)$ vanishes for $x = \xi$, where $a < \xi < b$.

Let $f'(a) > 0$ and $f'(b) < 0$. Then $f(x)$ is increasing in the neighbourhood of a and decreasing in the neighbourhood of b . The upper bound of $f(x)$ is therefore not attained at either a or b .

Now $f(x)$ being finitely derivable in (a, b) is continuous and bounded therein and its upper bound is attained at some point of (a, b) and this point being neither a nor b as shown above, it must be some point ξ between a and b . At this point $f'(x) = 0$. For otherwise, $f(x)$ would be increasing or decreasing in the neighbourhood of ξ according as $f'(\xi)$ is greater or less than zero, and so $f(\xi)$ would not be the upper bound of $f(x)$. $\therefore f'(\xi) = 0$.

Next let $\phi(x) = f(x) - Ax$, where $f'(a) < A < f'(b)$. Then $\phi(x)$ is finitely derivable in (a, b) . Now $\phi'(x) = f'(x) - A$. $\therefore \phi'(a)$ and $\phi'(b)$ are of opposite signs and hence, by what has been proved above, $\phi'(x) = 0$ at $x = \xi$, where $a < \xi < b$. Again $\phi'(x) = f'(x) - A$. $\therefore \phi'(\xi) = f'(\xi) - A = 0$, or, $f'(\xi) = A$, which proves the theorem.

Note. The property of passing through all intermediate values is a property of continuous functions. If however $\phi(x)$ is the derivative of a continuous function, $\phi(x)$ possesses this property, even though it is not necessarily continuous.

7.8. Lagrange's Mean Value Theorem.

If the function $f(x)$ is continuous in the closed interval (a, b) , and has a derivative, finite or infinite, at every point of the open interval (a, b) , then there exists a point ξ between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi), \quad (a < \xi < b).$$

Consider the function $F(x)$ defined by

$$F(x) = f(b) - f(x) - \frac{b-x}{b-a} \{f(b) - f(a)\}.$$

$F(x)$ is continuous in $a \leq x \leq b$ and has a derivative at every point of the interval $a < x < b$. Further $F(x)$ vanishes for $x=a$ and $x=b$. It follows from Rolle's theorem that there is a value ξ of x between a and b for which its derivative vanishes. But this derivative is

$$\frac{f(b) - f(a)}{b - a} - f'(x).$$

$$\therefore \frac{f(b) - f(a)}{b - a} = f'(\xi),$$

$$\text{or,} \quad f(b) = f(a) + (b-a) f'(\xi), \quad (a < \xi < b).$$

It should be observed that it has not been assumed in the proof that $f'(x)$ is continuous in (a, b) .

It is often convenient to express the theorem in the form

$$f(b) = f(a) + (b-a) f'\{a + \theta(b-a)\},$$

where θ is a number lying between 0 and 1. Of course $a + \theta(b-a)$ is merely another way of writing 'some number ξ between a and b '. If we put $b = a + h$, we obtain

$$f(a+h) = f(a) + h f'(a + \theta h), \quad 0 < \theta < 1,$$

which is also a form in which the theorem is very often stated.

Alternative Proof.

Consider the function

$$F(x) = f(x) - Ax,$$

where A is an undetermined constant. We choose A in such a way as to make $F(b)=F(a)$; then $f(b)-Ab=f(a)-Aa$,

$$\text{or, } A = \frac{f(b)-f(a)}{b-a}.$$

Now $F(x)$ is continuous in $a \leq x \leq b$, and derivable in $a < x < b$; further $F(a)=F(b)$. Therefore by Rolle's theorem (alternative form), we get at once

$$F'(\xi)=0, (a < \xi < b).$$

Now $F'(\xi)=f'(\xi)-A=0$, or $f'(\xi)=A$. Substituting the value of A , we get

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}, (a < \xi < b).$$

Note. $f(b)-f(a)$ is the increment of the function $f(x)$ corresponding to the increment $b-a$ of the variable x ; and this theorem giving the relation between the exact increment of $f(x)$ and that of the variable is also known as *the formula of finite increments of Lagrange*. It is also known as the Average value theorem or the theorem of the Mean.

Cor. 1. If the derivative of a function is zero at all points of an interval, the function is a constant throughout the interval.

We have $f(a+h)-f(a)=hf'(a+\theta h)$ $f'=0$. Therefore $f(a+h)=f(a)$ for values of h such that $a < a+h \leq b$; in other words, $f(x)$ is constant in (a, b) .

Cor. 2. Two functions, of which the derivatives are finite and equal in the interval (a, b) , differ from each other by a constant.

Let $F(x)=f(x)-\phi(x)$, where $f(x)$ and $\phi(x)$ are functions having equal and finite derivatives in (a, b) . Then the derivative of $F(x)$ is zero at all points of (a, b) and by Cor. 1, $F(x)=c$ in (a, b) , c being a constant, $\therefore f(x)=\phi(x)+c$.

Observation. The conditions of Rolle's theorem, and of the Mean Value theorem which depends upon it, are a set of *sufficient* conditions. They are by no means *necessary*, as the following example will show.

$$\text{Let } f(x) = \frac{1}{x} + \frac{1}{1-x} \text{ in } (0, 1).$$

Here (i) $f(x)$ is continuous in $(0 < x < 1)$ and not in $(0 \leq x \leq 1)$.

$$(ii) f'(x) = \frac{1}{(1-x)^2} - \frac{1}{x^2} \text{ exists in } 0 < x < 1;$$

$$(iii) f(0) \neq f(1), \text{ both being undefined.}$$

Thus the conditions of Rolle's theorem are not satisfied. But yet there exists a ξ (namely $\xi = \frac{1}{2}$), where $f'(\xi) = 0$, ($0 < \xi < 1$).

7.9. Cauchy's Mean Value Theorem.

If (i) the two functions $f(x)$ and $F(x)$ be continuous in $a \leq x \leq b$,

(ii) differentiable in $a < x < b$,

and (iii) $F'(x)$ does not vanish for any value of x in $a < x < b$, then there exists at least one value ξ of x such that

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)}, \quad (a < \xi < b).$$

Consider the function

$$\phi(x) = f(x) - A.F(x),$$

where A is chosen to make $\phi(a) = \phi(b)$. This requires that

$$f(a) - A.F(a) = f(b) - A.F(b),$$

$$\text{or, } f(b) - f(a) = A.[F(b) - F(a)].$$

Now $F(b) - F(a) \neq 0$, for otherwise $F'(x)$ would vanish at least once in $a < x < b$, which is contrary to hypothesis.

$$\therefore A = \frac{f(b) - f(a)}{F(b) - F(a)}.$$

By applying Rolle's theorem to $\phi(x)$, we get at once $\phi'(x) = 0$ for $x = \xi$ ($a < \xi < b$), i.e., $f'(x) - A.F'(x) = 0$ for $x = \xi$, or $f'(\xi) = A.F'(\xi)$. And since $F'(\xi) \neq 0$, we have

$$A = \frac{f'(\xi)}{F'(\xi)},$$

$$\text{or, } \frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)}, \quad (a < \xi < b).$$

Note. Lagrange's formula is only a particular case of Cauchy's formula and can be obtained from it by making $F(x) = x$. We shall indicate another way of establishing this result in the examples worked out.

Examples :

1. If (i) $f(x) = e^{-\frac{1}{x^2}}$ when $x \neq 0$ and $f(0) = 0$;
 (ii) $f(x) = x^2 \sin \frac{1}{x}$ when $x \neq 0$ and $f(0) = 0$, show that
 in both cases $f'(0)$ exists and is equal to 0. (C. H. 1968)

2. If $f(x) = x \tan^{-1} \frac{1}{x}$ when $x \neq 0$ and $f(0) = 0$, show that $f'(0)$ does not exist.

$$[Rf'(0) = \text{Lt}_{h \rightarrow 0} \frac{f(h)}{h} = \text{Lt}_{h \rightarrow 0} \tan^{-1} \frac{1}{h} = \frac{\pi}{2},$$

$$Lf'(0) = \text{Lt}_{h \rightarrow 0} \frac{f(-h)}{-h} = \text{Lt}_{h \rightarrow 0} \tan^{-1} \left(-\frac{1}{h} \right) = -\frac{\pi}{2}.$$

$\therefore Rf'(0) \neq Lf'(0)$, or $f'(0)$ does not exist.]

3. If $f(x) = \frac{x}{1+e^x}$ ($x \neq 0$) and $f(0) = 0$, show that $f'(0)$ does not exist.

4. Let $f(x) = (x-1)^{\frac{2}{3}} - 1$, so that $f(x) = 0$ when $x = 0$ and $x = 2$. Does $f'(x)$ vanish for any value of x between 0 and 2 ?

[Here $f'(x) = \frac{2}{3} \frac{1}{(x-1)^{\frac{1}{3}}}$ and this does not vanish for any value of

x between 0 and 2.

Evidently $f(x)$ does not possess a unique derivative at $x = 1$ and in the immediate neighbourhood of this point, $f'(x)$ oscillates infinitely. Hence Rolle's theorem is not applicable.]

5. A function $f(x)$ is defined as being equal to $1+x$ when $x \leq 0$, to x when $0 < x < 1$, to $2-x$ when $1 \leq x \leq 2$, and to $3x-x^2$ when $x > 2$. Discuss the continuity of $f(x)$ and the existence and continuity of $f'(x)$ for $x = 0$, $x = 1$, and $x = 2$. (C. U. 1966)

6. The functions u, v and their derivatives u', v' are continuous throughout a certain interval of values of x , and $uv' - u'v$ never vanishes at any point of the interval. Show that between any two roots of $u = 0$ lies one of $v = 0$ and conversely. Verify the theorem when $u = \cos x$, $v = \sin x$.

[If v does not vanish between two roots of $u = 0$, say u_1 and u_2 ,

then the function is continuous throughout the interval (u_1, u_2) and vanishes at its extremities. Hence $(u/v)' = (u'v - uv')/v^2$ must vanish between u_1 and u_2 , which contradicts our hypothesis.]

7. The formula $f(x+h) = f(x) + hf'(x+\theta h)$, $(0 < \theta < 1)$, is not true if $f(x) = \frac{1}{x}$ and $x < 0 < x+h$.

[For $f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} > 0$, as $\frac{1}{x+h} > 0$ and $\frac{1}{x} < 0$. Also $hf'(x+\theta h) = -\frac{h}{(x+\theta h)^2} < 0$. Hence $f(x+h) - f(x) \neq hf'(x+\theta h)$. The reason is that the conditions for the validity of the Mean Value theorem are not satisfied. As a matter of fact, the function $\frac{1}{x}$ is not continuous at $x=0$.]

8. If $x = -a$, $h = 2a$, $f(x) = x^{\frac{1}{3}}$, then the equation $f(x+h) = f(x) + hf'(x+\theta h)$ is satisfied by $\theta = \frac{1}{2} \pm \frac{\sqrt{3}}{18}$. (C. H. 1966)

[Although $f'(x)$ tends to ∞ as x tends to 0, there being a unique derivative at all points of the interval $(-a, a)$, the Mean Value theorem holds ; and we easily get the above value of θ .]

9. Apply the Mean Value theorem to find the derivative of a function of a function, assuming that the derivatives which occur are continuous. (C. U. 1968)

[Let $F\{f(x)\}$ be a function of a function of x . By definition, its derivative is $\lim_{h \rightarrow 0} \frac{F\{f(x+h)\} - F\{f(x)\}}{h}$. Now by the M. V. theorem, $f(x+h) = f(x) + hf'(\xi)$, $(x < \xi < x+h) = u+k$ say, so that $u = f(x)$ and $k = hf'(\xi)$.

Again $F(u+k) = F(u) + kF'(U)$, where $u < U < u+k$. As $h \rightarrow 0$, $\xi \rightarrow x$, also $k = hf'(\xi) \rightarrow 0$. Further as $k \rightarrow 0$, $U \rightarrow u$. Therefore

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{F\{f(x+h)\} - F\{f(x)\}}{h} &= \lim_{h \rightarrow 0} \frac{F(u+k) - F(u)}{h} = \lim_{h \rightarrow 0} \frac{kF'(U)}{h} \\ &= \lim_{h \rightarrow 0} \{f'(\xi) \cdot F'(U)\} = \lim_{h \rightarrow 0} \{f'(\xi)\} \cdot \lim_{h \rightarrow 0} \{F'(U)\} = f'(x) \cdot F'(u) \\ &= f(x) \cdot F'\{f(x)\}. \end{aligned}$$

10. If $f(x)$, $g(x)$, $h(x)$ be continuous in $a \leq x \leq b$ and differ-

entiable in $a < x < b$, there is a value ξ of x between a and b such that

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(\xi) & g'(\xi) & h'(\xi) \end{vmatrix} = 0.$$

[Consider the function

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix},$$

$F(x)$ is continuous in $a \leq x \leq b$ and differentiable in $a < x < b$, and vanishes for $x=a$ and $x=b$; therefore by Rolle's theorem, $F'(x)$ vanishes for $x=\xi$, where $a < \xi < b$; whence we easily get

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(\xi) & g'(\xi) & h'(\xi) \end{vmatrix} = 0, (a < \xi < b).$$

Note. If in the above we make $h(a)=h(b)=h(x)=1$ and hence $h'(x)=0$, we get at once Cauchy's Mean Value theorem. Again if we make $g(x)=x$ and $h(x)=1$, we immediately obtain the Mean Value theorem of Lagrange.]

11. If $f(x)$ and $g(x)$ be continuous in $a \leq x \leq b$ and differentiable in $a < x < b$, then there is a value ξ of x between a and b such that

$$\begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & g(a) \\ f'(\xi) & g'(\xi) \end{vmatrix}, (a < \xi < b).$$

[Consider the function

$$F(x) = \begin{vmatrix} f(x) & g(x) \\ f(b) & g(b) \end{vmatrix} - \frac{x-a}{b-a} \begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix}.$$

$F(x)$ is continuous in $a \leq x \leq b$ and differentiable in $a < x < b$; further it vanishes for $x=a$ and $x=b$. Therefore for a value ξ of x between a and b ,

$$F'(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} - \frac{1}{b-a} \begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix} = 0;$$

whence $\begin{vmatrix} f(a) & g(a) \\ f'(\xi) & g'(\xi) \end{vmatrix} = \frac{1}{b-a} \begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix}, a < \xi < b.$

$$\phi(a) = \phi(c) = \phi(b) = 0; \therefore \phi'(\xi_1) = 0, \phi'(\xi_2) = 0,$$

$$\text{where } a < \xi_1 < c < \xi_2 < b.$$

Again $\phi''(\xi) = 0$, where $\xi_1 < \xi < \xi_2$ and so $a < \xi < b$.]

16. Show that if $f(x)$ has derivatives of the first three orders, then $f(b) = f(a) + \frac{1}{2}(b-a)\{f'(a) + f'(b)\} - \frac{1}{6}(b-a)^3 f'''(c)$, where $a < c < b$.

[Consider the function

$$F(x) = f(x) - f(a) - \frac{1}{2}(x-a)\{f'(a) + f'(x)\}$$

$$- \left(\frac{b-a}{3} \right) [f(b) - f(a) - \frac{1}{2}(b-a)\{f'(a) + f'(b)\}].$$

$F(x)$ vanishes for $x = a$ and $x = b$; $\therefore F'(\xi) = 0$, ($a < \xi < b$).

$$\text{Now } F'(x) = \frac{1}{2}[f'(x) - f'(a) - (x-a)f''(x)]$$

$$- \frac{1}{6}(x-a)^2 \left\{ f(b) - f(a) - \frac{1}{2}(b-a)\{f'(a) + f'(b)\} \right\}$$

$$= \phi(x) \text{ say.}$$

$F'(x)$ or $\phi(x)$ vanishes for $x = \xi$ and also for $x = a$, $\therefore \phi'(x) = 0$ for $\phi = c$, where $a < c < \xi$;

$\phi'(c) = 0$ gives the required result.]

17. If $f(x)$, $g(x)$, $h(x)$ be continuous in $a \leq x \leq b$ and their

second derivatives exist in (a, b) , to prove that

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \frac{1}{6}(b-a)(c-a)(c-b) \begin{vmatrix} f''(a) & f''(\xi) & f''(b) \\ g''(a) & g''(\xi) & g''(b) \\ h''(a) & h''(\xi) & h''(b) \end{vmatrix}$$

where ξ and η lie between the least and greatest of a, b, c .

[Consider first the function

$$\phi(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix} - \frac{(x-a)(x-b)(c-a)(c-b)}{6}$$

and let $a < c < b$.

It is easy to see that $\phi(a) = \phi(b) = \phi(c) = 0$; $\therefore \phi'(c) = 0$, $\phi'(\eta_1) = 0$, $\phi'(\eta_2) = 0$, where $a < \eta_1 < c < \eta_2 < b$. As $\phi'(x) = 0$ for $x = \eta_1$ and $x = \eta_2$, where $x = 0$ for $x = \eta$, where $\eta_1 > \eta > \eta_2$. This gives at once

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \frac{1}{6}(c-a)(c-b) \begin{vmatrix} f''(a) & f''(b) & f''(c) \\ g''(a) & g''(b) & g''(c) \\ h''(a) & h''(b) & h''(c) \end{vmatrix} \quad \dots (1)$$

Otherwise.

Apply Lagrange's Mean Value theorem to

$$F(x) = \begin{vmatrix} f(a) & g(a) \\ f(x) & g(x) \end{vmatrix}$$

and the result follows at once.

Again it can be deduced from Ex. 10, by putting $h(x) = x - a$, so that $h(a) = 0$, $h(b) = b - a$ and $h'(x) = 1$.

12. Prove that $\lim_{h \rightarrow 0} \frac{f(a+h) + f(a-h)}{2h} = f'(a)$ if $f'(x)$ is continuous.

[$f(a+h) = f(a) + hf'(a + \theta_1 h)$, $f(a-h) = f(a) - hf'(a - \theta_2 h)$, where θ_1, θ_2 are positive proper fractions.

$$\therefore \lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{2h} = \lim_{h \rightarrow 0} \frac{f'(a + \theta_1 h) + f'(a - \theta_2 h)}{2} = f'(a),$$

since $f'(x)$ is continuous.]

13. If $f(x)$ and $\phi(x)$ are continuous in $a \leq x \leq b$, and differentiable in $a < x < b$ and $\phi'(x)$ and $\psi'(x)$ never vanish in (a, b) , then

$$\frac{f(\xi) - f(a)}{f'(\xi)} = \frac{\phi(\xi) - \phi(a)}{\phi'(\xi)} \quad (a < \xi < b)$$

[Apply the method of Ex. 10 to $F(x) = \{f(x) - f(a)\} \{\phi(b) - \phi(x)\}$.]

14. If $\psi''(x) \neq 0$ in $a < x < b$, then

$$\frac{\phi(b) - \phi(a) - (b-a)\phi'(a)}{\phi''(\xi)} = \frac{\psi(b) - \psi(a) - (b-a)\psi'(a)}{\psi''(\xi)} \quad (a < \xi < b).$$

[Apply the same method to $F(x)$ where

$$F(x) = \phi(b) - \phi(x) - (b-x)\phi'(x)$$

$$- \frac{\phi(b) - \phi(a) - (b-a)\phi'(a)}{\phi''(\xi)} \{\psi(b) - \psi(a) - (b-a)\psi'(a)\} \{\psi(b) - \psi(x) - (b-x)\psi'(\xi)\}$$

15. Assuming $f'''(x)$ continuous in $a \leq x \leq b$, show that

$$f(c) - f(a) - \frac{b-a}{c-a} f'(a) - \frac{1}{2} (c-a) \frac{b-a}{c-a} f''(\xi),$$

where c and ξ both lie in (a, b) .

[Hint. Consider

$$\phi(x) = f(x) + f(a) \frac{x-a}{x-c} - f(c) \frac{c-a}{x-a} - \left\{ f(b) + f(a) \frac{b-a}{b-c} - f(c) \frac{c-a}{b-a} \right\} \frac{(b-a)(b-c)}{(x-a)(x-c)}$$

From (1) and (2), we get

$$\phi'(x_1) + \phi(x_2) - 2\phi\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}(x_2 - x_1) [\phi'(\xi_2) - \phi'(\xi_1)] \dots \quad (3)$$

Applying the Mean Value theorem to $\phi'(x)$ for the interval (ξ_1, ξ_2) , we see that there is a number ξ ($\xi_1 < \xi < \xi_2$) such that

$$\phi'(\xi_2) - \phi'(\xi_1) = (\xi_2 - \xi_1) \phi''(\xi) \dots \dots \quad (4)$$

From (3) and (4), we get

$$\phi(x_1) + \phi(x_2) - 2\phi\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}(x_2 - x_1) (\xi_2 - \xi_1) \phi''(\xi).$$

Since the right-hand side is not negative, we have

$$\phi(x_1) + \phi(x_2) \geq 2\phi\left(\frac{x_1 + x_2}{2}\right),$$

$$\text{or,} \quad \phi\left[\frac{1}{2}(x_1 + x_2)\right] \leq \frac{1}{2}[\phi(x_1) + \phi(x_2)].$$

(ii) We have just proved that

$$\phi\left[\frac{1}{2}(x_1 + x_2)\right] \leq \frac{1}{2}[\phi(x_1) + \phi(x_2)].$$

As $x_1, x_2, x_3, \dots, x_n$ all lie in (a, b) , we similarly get

$$\phi\left[\frac{1}{2}(x_3 + x_4)\right] \leq \frac{1}{2}[\phi(x_3) + \phi(x_4)].$$

Adding $\frac{1}{2}[\phi(x_1) + \phi(x_2) + \phi(x_3) + \phi(x_4)] \geq \phi\left[\frac{1}{2}(x_1 + x_2)\right] + \phi\left[\frac{1}{2}(x_3 + x_4)\right]$

$$\geq 2 \cdot \phi\left[\frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}}{2}\right] = 2\phi\left[\frac{1}{2^2}(x_1 + x_2 + x_3 + x_4)\right]$$

$$\text{i.e., } \phi\left[\frac{1}{2^2}(x_1 + x_2 + x_3 + x_4)\right] \leq \frac{1}{2^2}[\phi(x_1) + \phi(x_2) + \phi(x_3) + \phi(x_4)].$$

By a repeated application of this process, we get

$$\phi\left[\frac{1}{n}(x_1 + x_2 + \dots + x_n)\right] \leq \frac{1}{n}[\phi(x_1) + \phi(x_2) + \dots + \phi(x_n)]$$

where n is a power of 2.

If n is not a power of 2, we consider the n values $x_1, x_2, \dots, x_n$ and p values each equal to $\frac{1}{n}(x_1 + x_2 + \dots + x_n)$ such that $n+p$ is a power of 2. Further $\frac{1}{n}(x_1 + x_2 + \dots + x_n)$ lies between the greatest and least of $x_1, x_2, \dots, x_n$ and therefore lies in (a, b) .

Applying the result (1) to these $n+p$ quantities, we get

$$\begin{aligned} & \phi \left[\frac{1}{n+p} \{x_1 + x_2 + \cdots + x_n + \frac{p}{n}(x_1 + x_2 + \cdots + x_n)\} \right] \\ & \leq \frac{1}{n+p} \left[\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n) + p \cdot \phi \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right) \right] \end{aligned}$$

$$\text{or, } \phi \left[\frac{1}{n} (x_1 + x_2 + \cdots + x_n) \right] \leq \frac{1}{n+p} [\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n)]$$

$$+ \frac{p}{n+p} \phi \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right),$$

$$\text{or } \left(1 - \frac{p}{n+p} \right) \phi \left[\frac{1}{n} (x_1 + x_2 + \cdots + x_n) \right] \leq \frac{1}{n+p} [\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n)]$$

$$\text{i.e., } \phi \left[\frac{1}{n} (x_1 + x_2 + \cdots + x_n) \right] \leq \frac{1}{n} [\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n)].$$

CHAPTER 8

EXPANSIONS

8.1. Mean Value theorem of the second order.

If $f(x)$ and $f'(x)$ be continuous in the closed interval (a, b) and $f''(x)$ exists in the open interval (a, b) , then there is a value ξ of x between a and b for which

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(\xi).$$

Consider the function

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \left(\frac{b-x}{b-a}\right)^2 \{f(b) - f(a) - (b-a)f'(a)\}.$$

Under the given hypotheses, $F(x)$ is continuous in $a \leq x \leq b$, and $F'(x)$ exists in $a < x < b$; also $F(x)$ vanishes when $x=a$ and $x=b$. Therefore by Rolle's theorem, $F'(x)$ vanishes, when $x=\xi$ ($a < \xi < b$).

Now

$$F'(x) = \frac{2(b-x)}{(b-a)^2} \left\{ f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!} f''(x) \right\}$$

and as $b-x \neq 0$ for $x=\xi$, we have

$$F'(\xi) = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!} f''(\xi) = 0;$$

$$\therefore f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(\xi), \quad (a < \xi < b). \quad \dots (1)$$

A number ξ between a and b can be expressed as $a + \theta_2(b-a)$ where $0 < \theta_2 < 1$ and so (1) can be written in the form

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''\{a + \theta_2(b-a)\},$$

$$(0 < \theta_2 < 1) \quad \dots \quad \dots \quad (2)$$

Further on putting $b=a+h$, the same can be written as

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a + \theta_2 h), \quad (0 < \theta_2 < 1). \quad \dots (3)$$

This is the standard form of what may be called *the mean value theorem of the second order*.

Note 1. Observe that the continuity of $f'(x)$ in $a \leq x \leq b$ implies the continuity of $f(x)$ in the same closed interval.

Note 2. If the second derivative $f''(x)$ be continuous in $a \leq x \leq b$, [implying of course the continuity of $f(x)$ and $f'(x)$ in the same closed interval] and the third derivative $f'''(x)$ exists in $a < x < b$, it can be proved as above that there exists a value ξ of x between a and b such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(\xi),$$

or putting $b = a + h$,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a + \theta_3 h), \quad (0 < \theta_3 < 1).$$

8.2. Taylor's theorem or the General Mean Value theorem.

If (i) the $(n-1)$ th derivative $f^{(n-1)}(x)$ be continuous in $a \leq x \leq b$, and (ii) the n th derivative $f^{(n)}(x)$ exists in $a < x < b$ there exists a value ξ of x between a and b such that

$$\begin{aligned} f(b) = & f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots \\ & + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(\xi), \end{aligned}$$

or putting $b = a + h$,

$$\begin{aligned} f(a+h) = & f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots \\ & + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta_n h), \quad (0 < \theta_n < 1). \end{aligned}$$

First note that the condition (i) ensures the continuity of $f(x)$, $f'(x)$, $f''(x)$, ..., $f^{(n-2)}(x)$ in the closed interval (a, b) .

Consider the function

$$\begin{aligned} F(x) = & f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!} f''(x) - \dots \\ & - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) - \left(\frac{b-x}{b-a} \right)^n [f(b) - f(a) \\ & - (b-a)f'(a) - \frac{(b-a)^2}{2!} f''(a) - \dots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a)], \end{aligned}$$

or putting
$$\phi(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \dots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x),$$

we get
$$F(x) = \phi(x) - \left(\frac{b-x}{b-a}\right)^n \phi(a).$$

Under the given hypotheses, $F(x)$ is continuous in $a \leq x \leq b$ and $F'(x)$ exists in $a < x < b$; further $F(x)$ vanishes for $x=a$ and $x=b$; hence by Rolle's theorem, $F'(x)$ vanishes for a value ξ of x between a and b .

Now
$$F'(x) = \frac{n(b-x)^{n-1}}{(b-a)^n} \left\{ \phi(a) - \frac{(b-a)^n}{n!} f^{(n)}(x) \right\}$$
 and since $b-x \neq 0$, when $x=\xi$,

we get
$$\phi(a) - \frac{(b-a)^n}{n!} f^{(n)}(\xi) = 0,$$

i.e.,
$$f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \dots - \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{(b-a)^n}{n!}f^{(n)}(\xi) = 0,$$

or,
$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(\xi),$$

where $a < \xi < b$ (1)

Putting $b=a+h$, so that $\xi=a+\theta_n h$ where $0 < \theta_n < 1$,

we get
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta_n h), \dots \quad (2)$$

The above expression for $f(a+h)$ is a finite series of $n+1$ terms and is called *Taylor's series with a remainder*; the last term $R_n = \frac{h^n}{n!}f^{(n)}(a+\theta_n h)$, $0 < \theta_n < 1$, is called *the remainder after n terms in Taylor's series*. The form of the remainder given above is due to Lagrange. The remainder can however be expressed in other forms which we next proceed to find. In what follows we shall put θ for θ_n .

8'3. Other forms of the remainder.

Let $f^{(n-1)}(x)$ be continuous in $a \leq x \leq b$ and let $f^{(n)}(x)$ exist in $a < x < b$.

Let $f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$, where $R_n = (b-a)^p P$. Consider the function

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \dots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) - (b-x)^p P,$$

where P is determined so as to make $F(a) = 0$; this gives

$$(b-a)^p P = f(b) - f(a) - (b-a)f'(a) - \dots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) \dots (1)$$

Under the given conditions $F(x)$ is continuous in $a \leq x \leq b$ and $F'(x)$ exists in $a < x < b$; also $F(a) = 0$ in virtue of (1) and obviously $F(b) = 0$. Therefore for a value ξ of x between a and b , $F'(\xi) = 0$.

$$\text{Now } F'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) + p(b-x)^{p-1} P.$$

Hence $F'(\xi) = 0$ gives

$$P = \frac{(b-\xi)^{n-p}}{(n-1)! p} f^{(n)}(\xi);$$

or, putting $b = a + h$, and $\xi = a + \theta h$ where $0 < \theta < 1$, we have

$$P = \frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(a + \theta h).$$

$$\therefore R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(a + \theta h), \quad 0 < \theta < 1.$$

This form of the remainder in Taylor's expansion of $f(a+h)$ is due to Schlömilch and Roche. $p = n$ in this gives Lagrange's form of the remainder. On making $p = 1$ we get Cauchy's form which is

$$R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a + \theta h).$$

8'4. Maclaurin's theorem.

If in Taylor's theorem we put $a = 0$ and $h = x$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n,$$

where
$$R_n = \frac{x^n(1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(\theta x), \quad (\text{Schlömilch}),$$

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x), \quad (\text{Cauchy}),$$

$$= \frac{x^n}{n!} f^{(n)}(\theta x). \quad (\text{Lagrange}).$$

The formula assumes that $f^{(n-1)}(x)$ is continuous in $(0, x)$ and $f^{(n)}(x)$ exists in the interval and the number θ lies between 0 and 1. Of course, θ is not the same number in the three different forms of the remainder.

8'5. Taylor's infinite Series.

If $f(x)$ is a function all of whose derivatives are continuous in an interval $(a, a+\eta)$ and if h is less than η , we have

$$f(a+h) = f(a) + hf'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h),$$

where $0 < \theta < 1$, for all values of n . Or if we put

$$S_n = \sum_{p=1}^{n-1} \frac{h^p}{p!} f^{(p)}(a), \quad R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h),$$

we have

$$f(a+h) - S_n = R_n.$$

If in addition R_n tends to 0 as n tends to infinity, then

$$f(a+h) = \lim_{n \rightarrow \infty} S_n = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots \text{to infinity},$$

which is Taylor's infinite series.

Maclaurin's infinite series

When the derivatives of $f(x)$ up to any order exist and are continuous in $(0, x)$ and R_n can be made to approach 0 by making n indefinitely large, we get Maclaurin's infinite series:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots \text{to infinity}.$$

Note 1. The continuity of the derivatives of $f(x)$ is not sufficient for the validity of Taylor's (or Maclaurin's) series. A direct discussion of the behaviour of R_n is always essential.

Note 2. $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$ is a series of the

type

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

and is called a *power series* in x .

$f(x)$ is representable by a power series if

- (i) $f(x)$ has derivatives of all orders.
- (ii) Maclaurin's series corresponding to $f(x)$ is convergent.
- (iii) The remainder $R_n \rightarrow 0$ as $n \rightarrow \infty$.

Functions which can be represented by power series are called *Analytic* functions.

8.6. Maclaurin's expansions of :

$$e^x, a^x, \sin x, \cos x, \log(1+x), (1+x)^m.$$

I. $f(x) = e^x$. We have $f(x) = f'(x) = f''(x) = \dots = f^{(n)}(x) = e^x$. Hence $f(x)$ and its derivatives of all orders are continuous in $(0, x)$. By Maclaurin's theorem, using Lagrange's form of the remainder,

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x}, \quad (0 < \theta < 1) \quad \dots (1)$$

Now $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ and $e^{\theta x} < e^x =$ a definite finite quantity.

$\therefore R_n = \frac{x^n}{n!} e^{\theta x} \rightarrow 0$ as $n \rightarrow \infty$. We thus have

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \quad \dots \quad \dots \quad (2)$$

In (1) we put $x=1$; we get

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-1)!} + \frac{e^\theta}{n!}, \quad (0 < \theta < 1).$$

Since $\frac{e^\theta}{n!} < \frac{e}{n!}$ ($0 < \theta < 1$) $\rightarrow 0$ as $n \rightarrow \infty$, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots \quad \dots \quad \dots \quad (3)$$

From (2) and (3), it follows

$$\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots\right)^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

a result which is commonly known as the *Exponential Theorem*.

II. $f(x) = a^x$. Here $f'(x) = a^x \log a$, $f''(x) = a^x (\log a)^2, \dots$,
 $f^{(n)}(x) = a^x (\log a)^n$.

$$\text{Hence } a^x = 1 + \frac{x \log a}{1!} + \frac{(x \log a)^2}{2!} + \dots + \frac{(x \log a)^{n-1}}{(n-1)!} \\ + \frac{(x \log a)^n}{n!} a^{\theta x}, \quad (0 < \theta < 1).$$

As above $\text{Lt } \frac{(x \log a)^n}{n!} = 0$ and $a^{\theta x} < a^x = a$ a definite finite quantity.

$$\therefore R_n = \frac{(x \log a)^n}{n!} a^{\theta x} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ and we have}$$

$$a^x = 1 + \frac{x \log a}{1!} + \frac{x^2 (\log a)^2}{2!} + \dots + \frac{x^n (\log a)^n}{n!} + \dots$$

III. $f(x) = \sin x$. We have $f^{(n)}(x) = \sin \left(x + n \frac{\pi}{2} \right)$. For $x=0$, the values of $f(x)$ and its successive derivatives form a recurring sequence of the values 0, 1, 0, -1. Now

$$R_n = \frac{x^n}{n!} \sin \left(\theta x + \frac{n\pi}{2} \right)$$

$$\text{and since } \left| R_n \right| = \left| \frac{x^n}{n!} \right| \cdot \left| \sin \left(\theta x + \frac{n\pi}{2} \right) \right| \leq \left| \frac{x^n}{n!} \right|,$$

we see that $R_n \rightarrow 0$ as $n \rightarrow \infty$ for every value of x .

Now $f^{(n)}(0) = \sin \frac{n\pi}{2}$ and we get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \dots$$

IV. $f(x) = \cos x$. Proceeding exactly as with $\sin x$, we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

V. $f(x) = \log(1+x)$. In the first place we note that if $1+x \leq 0$, $f(x)$ is undetermined or unreal. Hence if $1+x > 0$, i.e., $x > -1$, $f(x) = \log(1+x)$ possesses derivatives of every order. Now

$$f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{(1+x)^n}, \quad (x > -1).$$

If R_n denotes Lagrange's form of the remainder,

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{(-1)^{n-1}}{n} \cdot \left(\frac{x}{1+\theta x} \right)^n.$$

(i) Let $0 \leq x \leq 1$, so that $\frac{x}{1+\theta x}$ and therefore $\left(\frac{x}{1+\theta x} \right)^n$ is positive and < 1 , whatever value n may have. Also $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$. We see that as $n \rightarrow \infty$, $R_n \rightarrow 0$. Thus $R_n \rightarrow 0$ as $n \rightarrow \infty$, when $0 \leq x \leq 1$.

(ii) Let $-1 < x < 0$. In this case $\frac{x}{1+\theta x}$ may not be numerically less than unity and we fail to draw any definite conclusion from Lagrange's form of the remainder. Taking Cauchy's form

$$\begin{aligned} R_n &= \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta x) \\ &= (-1)^{n-1} \cdot x^n \cdot \frac{1}{1+\theta x} \cdot \left(\frac{1-\theta}{1+\theta x} \right)^{n-1}. \end{aligned}$$

Since $|x| < 1$, $0 < \frac{1-\theta}{1+\theta x} < 1$ and so $0 < \left(\frac{1-\theta}{1+\theta x} \right)^{n-1} < 1$; also $\frac{1}{1+\theta x} < \frac{1}{1-|x|}$, and $x^n \rightarrow 0$ as $n \rightarrow \infty$. Hence $R_n \rightarrow 0$ as $n \rightarrow \infty$. Thus the conditions for the Maclaurin's expansion of $\log(1+x)$ are satisfied for $-1 < x \leq 1$; also

$$f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!.$$

So we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n!} + \dots$$

VI. $f(x) = (1+x)^m$; m any real number. We have

$$f^{(n)}(x) = m(m-1)(m-2)\dots(m-n+1)(1+x)^{m-n}.$$

If m is a positive integer, $f^{(m+1)}(x)$ and the successive derivatives that follow all vanish for every value of x . Hence when m is a positive integer

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \dots + x^m,$$

for every value of x . This is Newton's Binomial theorem.

If m is any real number other than a positive integer, $(1+x)^m$ has continuous derivatives of every order when $1+x > 0$, i.e., for $x > -1$.

If R_n denotes Cauchy's form of remainder,

$$\begin{aligned} R_n &= \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta x) \\ &= \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n \left(\frac{1-\theta}{1+\theta x} \right)^{n-1} (1+\theta x)^{m-1} \end{aligned}$$

Let $-1 < x < 1$, i.e., $|x| < 1$. Since $0 < \theta < 1$, we have

$$0 < 1-\theta < 1+\theta x,$$

$$\text{or, } 0 < \frac{1-\theta}{1+\theta x} < 1,$$

$$\text{or, } 0 < \left(\frac{1-\theta}{1+\theta x} \right)^{n-1} < 1.$$

(i) Let $m-1$ be positive,

$$\text{Now } 0 < 1+\theta x < 1+1=2,$$

$$\therefore 0 < (1+\theta x)^{m-1} < 2^{m-1}.$$

(ii) Let $m-1$ be negative.

$$\text{Since } \theta x \geq -|x|,$$

$$1+\theta x \geq 1-|x|,$$

$$\therefore (1+\theta x)^{m-1} < (1-|x|)^{m-1}$$

Thus $(1+\theta x)^{m-1}$ is always bounded.

$$\text{Also } \lim_{n \rightarrow \infty} \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n = 0.$$

$$\therefore R_n = \frac{m(m-1)\cdots(m-n+1)}{(n-1)!} x^n (1+\theta x)^{m-1} \left(\frac{1-\theta}{1+\theta x} \right)^{n-1}$$

$$= 0 \text{ as } n \rightarrow \infty \text{ when } |x| < 1.$$

The conditions for the Maclaurin's expansion of $(1+x)^m$ are satisfied. Hence

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \dots,$$

$$\text{when } |x| < 1.$$

87. Taylor's theorem deduced from Cauchy's Mean Value theorem.

Taylor's theorem has been obtained as the general mean value theorem of Lagrange. It may also be obtained from Cauchy's mean value theorem thus :

$$\text{Put } f(a+h) = f(a) + hf'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \psi(h),$$

$$\text{and } \frac{h^n}{n!} = \phi(h).$$

It is easy to see that both $\psi(h)$ and $\phi(h)$ and their first $n-1$ derivatives vanish for $h=0$; also $\phi^{(n)}(h)=1$.

$$\begin{aligned} \text{Now } \frac{\psi(h)}{\phi(h)} &= \frac{\psi(h) - \psi(0)}{\phi(h) - \phi(0)} \\ &= \frac{\psi'(h_1)}{\phi'(h_1)}, \quad (0 < h_1 < h), \\ &= \frac{\psi'(h_1) - \psi'(0)}{\phi'(h_1) - \phi'(0)} \\ &= \frac{\psi''(h_2)}{\phi''(h_2)}, \quad (0 < h_2 < h_1), \\ &\dots \dots \dots \dots \dots \\ &= \frac{\psi^{(n)}(h_n)}{\phi^{(n)}(h_n)}, \quad (0 < h_n < h_{n-1}). \end{aligned}$$

$$\text{Thus } \psi(h) = \phi(h) \cdot \frac{\psi^{(n)}(\theta h)}{\phi^{(n)}(\theta h)}, \quad (0 < \theta < 1).$$

Since $\phi^{(n)}(\theta h)=1$, we get

$$R_n = \psi(h) = \frac{h^n}{n!} f^{(n)}(a + \theta h).$$

88. Young's form of Taylor's theorem.

If $f(x)$ has n derivatives at $x=a$, then

$$f(a+h) = f(a) + hf'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} \{f^{(n)}(a) + \eta\},$$

where $\eta \rightarrow 0$ with h .

[See Art. 11'9.]

Examples :

1. (i) If $f(x) = (x-1)e^x + 1$, show that $f(x)$ is positive for all positive values of x .

(ii) Show that $x - \sin x$ is an increasing function of x for any interval of values of x . (C. H. 1964)

2. Show that $\frac{x}{\sin x}$ steadily increases in $0 < x < \frac{\pi}{2}$; hence or otherwise show that

$$\frac{2x}{\pi} < \sin x < x \text{ if } 0 < x < \frac{\pi}{2}. \quad (\text{C. H. 1961})$$

3. If $0 < x < 1$, prove that

$$x < \log \frac{1}{1-x} < \frac{x}{1-x}. \quad (\text{C. H. 1963})$$

[Hint. Let $F(x) = \frac{x}{1-x} - \log \frac{1}{1-x}$; then $F'(x) = \frac{x}{(1-x)^2} > 0$ if $x > 0$.

Also $F(0) = 0$. $\therefore F(x) > F(0) = 0$; whence $\frac{x}{1-x} > \log \frac{1}{1-x}$.

Again let $\phi(x) = \log \frac{1}{1-x} - x$; then $\phi'(x) = \frac{x}{1-x} > 0$ if $1 > x > 0$.

Also $\phi(0) = 0$. $\therefore \phi(x) > 0$ if $0 < x < 1$, or $\log \frac{1}{1-x} > x$, if $1 > x > 2$.]

4. If x be positive, apply the mean value theorem to show that

$$(i) \ x > \log(1+x) > x - \frac{x^2}{2}. \quad (\text{C. H. 1960})$$

$$(ii) \ 1 > \cos x > 1 - \frac{1}{2}x^2.$$

5. If $0 < x < \frac{\pi}{2}$, show that $x > \sin x > x - \frac{x^3}{6}$ and $\tan x > x + \frac{x^3}{3}$.

6. If x is small and positive, $(1+x)^{-\frac{1}{2}}$ is approximately equal to $1 - \frac{1}{2}x$ with an error which is less than $\frac{3}{8}x^2$.

7. If $f'(x)$ exists and $\neq 0$, evaluate the following limits:

$$(i) \ \lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h}; \quad (ii) \ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{f(x-h) - f(x)}.$$

8. Prove that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$$

if $f''(x)$ exists.

(C. H. 1966)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x + \theta_1 h), \quad (0 < \theta_1 < 1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x - \theta_2 h), \quad (0 < \theta_2 < 1)$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \frac{1}{2} [f''(x+0) + f''(x-0)] = f''(x),$$

since $f''(x)$ exists.]

9. In the Mean Value theorem

$$f(h) = f(0) + hf'(\theta h), \quad 0 < \theta < 1,$$

show that the limiting value of θ as $h \rightarrow 0$ is $\frac{1}{2}$ or $\frac{1}{\sqrt{3}}$ according as $f(x) = \cos x$ or $\sin x$.

10. If $f^{(n+1)}(x)$ is continuous and $\neq 0$, the number θ which occurs in the Lagrange form of remainder of Taylor's theorem,

viz., $\frac{h^n}{n!} f^{(n)}(x + \theta h)$, tends to the limit $\frac{1}{n+1}$ as $h \rightarrow 0$.

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \dots + \frac{h^n}{n!} f^{(n)}(x + \theta h) \\ &= f(x) + hf'(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x + \theta' h), \end{aligned}$$

where both θ and θ' lie between 0 and 1. Hence

$$f^{(n)}(x + \theta h) = f^{(n)}(x) + \frac{hf^{(n+1)}(x + \theta' h)}{n+1}.$$

Again by the mean value theorem,

$$f^{(n)}(x + \theta h) = f^{(n)}(x) + \theta h f^{(n+1)}(x + \theta\theta'' h), \quad (0 < \theta'' < 1),$$

$$\therefore \theta f^{(n+1)}(x + \theta\theta'' h) = \frac{f^{(n+1)}(x + \theta' h)}{n+1}.$$

Let $h \rightarrow 0$, then

$$\lim_{h \rightarrow 0} \theta f^{(n+1)}(x) = \frac{f^{(n+1)}(x)}{n+1}, \text{ as } f^{(n+1)}(x) \text{ is continuous. Further}$$

as $f^{(n+1)}(x) \neq 0$, we get

$$\lim_{h \rightarrow 0} \theta = \frac{1}{n+1}. \quad]$$

11. If $a < c < b$ and $f''(x)$ exists everywhere in (a, b) , prove that

$$\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-a)(c-b)} = \frac{1}{2}f''(\xi),$$

where $a < \xi < b$. Deduce that

$$f(x+h) - 2f(x) + f(x-h) = h^2 f''(\xi),$$

where ξ lies between $x-h$ and $x+h$.

[Consider the function

$$\phi(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) + \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c) - f(x);$$

$$\text{then } \phi'(x) = \frac{2x-(b+c)}{(a-b)(a-c)}f(a) + \frac{2x-(c+a)}{(b-c)(b-a)}f(b) + \frac{2x-(a+b)}{(c-a)(c-b)}f(c) - f(x),$$

$$\text{and } \phi''(x) = \frac{2}{(a-b)(a-c)}f(a) + \frac{2}{(b-c)(b-a)}f(b) + \frac{2}{(c-a)(c-b)}f(c) - f''(x).$$

Now $\phi(x)=0$ for $x=a$ and $x=c$; therefore by Rolle's theorem, $\phi'(x)=0$ for $x=\xi_1$, where $a < \xi_1 < c$.

Again $\phi(x)=0$ for $x=c$ and $x=b$; so $\phi'(x)=0$ for $x=\xi_2$, where $c < \xi_2 < b$.

Hence $\phi''(x)=0$ for $x=\xi$, where $\xi_1 < \xi < \xi_2$, i.e., $a < \xi < b$.

This gives at once

$$\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-a)(c-b)} = \frac{1}{2}f''(\xi).$$

Putting $a=x-h$, $c=x$ and $b=x+h$, we obtain

$$f(x+h) - 2f(x) + f(x-h) = h^2 f''(\xi),$$

where $x-h < \xi < x+h$.]

12. Justify the Maclaurin's expansion of

- (i) $e^{ax} \sin bx$,
- (ii) $e^{ax} \cos (bx+d)$,
- (iii) $\arctan x$.

13. Show that $\log x$ and $\cot x$ cannot be expanded as Maclaurin's series.

14. Show that

$$\frac{2}{2x+1} < \log \frac{x+1}{x} < \frac{2x+1}{2x(x+1)} \quad (x > 0).$$

(*Math. Tripos 1931, 1933, 1936*)

15. If n be positive, show that $\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0$ and deduce that

$$\lim_{x \rightarrow 0} \frac{\frac{1}{x^2}}{x^n} = 0.$$

[First let n be a positive integer, then

$$\frac{x^n}{e^x} = \frac{x^n}{1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}e^{\theta x}} = \frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\dots+\frac{e^{\theta x}}{n!}}.$$

Now n being fixed and $e^{\theta x} \rightarrow \infty$ as $x \rightarrow \infty$, we have

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0. \quad \dots \quad (1)$$

Next let $m < n < m+1$, where m is an integer. Then $\frac{x^n}{e^x}$ li

between $\frac{x^m}{e^x}$ and $\frac{x^{m+1}}{e^x}$, both of which tend to zero as x tends to ∞ .

$$\therefore \lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0.$$

Now put $x^2 = \frac{1}{y}$, then

$$\lim_{x \rightarrow 0} \frac{\frac{1}{x^2}}{x^n} = \lim_{y \rightarrow \infty} \frac{y^{\frac{n}{2}}}{e^{\frac{1}{y}}} = 0, \text{ by (1).}$$

It follows that $\lim_{x \rightarrow 0} \left\{ A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \dots + \frac{A_n}{x^n} \right\} \frac{1}{e^{\frac{1}{x^2}}} = 0$.

Thus $\frac{1}{e^{\frac{1}{x^2}}} \times \left(\text{a polynomial in } \frac{1}{x} \right) \rightarrow 0$ as $x \rightarrow 0$.]

16. Show that the function defined by

$$f(x) = e^{-\frac{1}{x^2}} \quad \text{when } x \neq 0 \text{ and } f(0) = 0$$

possesses continuous derivatives for every value of x but it cannot be expanded as a Maclaurin's series.

[We have

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x} = 0. \quad (\text{Ex. 15})$$

$$f'(x) = \frac{2e^{-\frac{1}{x^2}}}{x^3}, \quad \text{when } x \neq 0.$$

$$\therefore f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \rightarrow 0} \frac{2e^{-\frac{1}{x^2}}}{x^4} = 0.$$

$$f''(x) = \left(\frac{4}{x^5} - \frac{6}{x^3} \right) e^{-\frac{1}{x^2}}, \quad \text{when } x \neq 0.$$

$$f''(0) = \lim_{x \rightarrow 0} \left\{ \left(\frac{4}{x^5} - \frac{6}{x^3} \right) e^{-\frac{1}{x^2}} \right\} = 0.$$

Proceeding thus, we can prove that

$$f^{(n)}(0) = \lim_{x \rightarrow 0} \left\{ \left(\text{a polynomial in } \frac{1}{x} \right) \cdot e^{-\frac{1}{x^2}} \right\} = 0.$$

The function defined above has derivatives of all orders at $x=0$, and their values are

$$f'(0) = f''(0) = f'''(0) = \dots = f^{(n)}(0) = \dots = 0.$$

Hence the Maclaurin's series for the function is

$$\begin{aligned} f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots \\ = 0 + 0 + 0 + \dots + 0 + \dots = 0, \end{aligned}$$

which is convergent having a sum equal to 0; but this is certainly not equal to $f(x)$ for any value of x other than 0. This example is due to Cauchy and the function is known as the Cauchy function. It is not an analytic function.]

CHAPTER 9

MAXIMA & MINIMA : INDETERMINATE FORMS

9.1. Maximum and minimum value.

The value $f(\xi)$ assumed by $f(x)$ at $x=\xi$ is a *maximum* value of $f(x)$ if for any value of x in the neighbourhood $0 < |x - \xi| < \delta$ of ξ , $f(x) < f(\xi)$; or putting $x - \xi = h$, $f(\xi + h) < f(\xi)$ when $0 < |h| < \delta$. In other words, $f(\xi)$ is a maximum value of $f(x)$ if $f(\xi + h) - f(\xi)$ is negative whether h be positive or negative provided it is numerically less than δ .

Again $f(\xi)$ is a *minimum* value of $f(x)$ if for any value of x in the neighbourhood $0 < |x - \xi| < \delta$ of ξ , $f(x) > f(\xi)$, or if $x - \xi = h$, $f(\xi + h) > f(\xi)$ when $0 < |h| < \delta$. Stated otherwise $f(\xi)$ is a minimum value of $f(x)$ if $f(\xi + h) - f(\xi)$ is positive, whether h be positive or negative but numerically less than δ .

The common name for a maximum or a minimum is an *extremum*.

9.2. Theorem. A necessary condition for a maximum or minimum value of $f(x)$ at $x=\xi$ is $f'(\xi)=0$, if it exists.

We suppose $f'(x)$ exists at $x=\xi$ and is continuous. Let $f(x)$ be a maximum at $x=\xi$. It is easy to see that in the left-neighbourhood of ξ , $f(x)$ is an increasing function and so $f'(x)$ is positive. Again in the right-neighbourhood of ξ , $f(x)$ is a decreasing function and so $f'(x)$ is negative. Thus as x passes through ξ , $f'(x)$ changes from a positive value to a negative value and being continuous, it must pass through a zero-value at $x=\xi$. $\therefore f'(\xi)=0$.

Next let $f(x)$ be a minimum at $x=\xi$. In the neighbourhood to the left of ξ , $f(x)$ is decreasing and hence $f'(x) < 0$, whereas in the neighbourhood to the right of ξ , $f(x)$ is increasing so that $f'(x) > 0$. Thus as x passes through ξ , $f'(x)$ changes from a negative value to a positive value and therefore $f'(x)=0$ at $x=\xi$.

Thus if $f(\xi)$ be an extremum value of $f(x)$, $f'(\xi)=0$.

To sum up, if the sign of $f'(x)$ changes at $x=\xi$ from positive to

negative, then $f(\xi)$ is a maximum value of $f(x)$ and if it changes from negative to positive, then $f(\xi)$ is a minimum value.

Alternative Proof.

Suppose $f(\xi)$ is a maximum. Let $0 < h < \delta$;
then $f(\xi \pm h) - f(\xi) < 0$.

$$\therefore \{f(\xi+h)-f(\xi)\}/h < 0. \text{ So } \lim_{h \rightarrow 0} \frac{f(\xi+h)-f(\xi)}{h} \leq 0,$$

$$\text{or } Rf'(\xi) \leq 0. \text{ Again } \{f(\xi-h)-f(\xi)\}/(-h) > 0$$

$$\text{and so } \lim_{h \rightarrow 0} \frac{f(\xi-h)-f(\xi)}{-h} \geq 0, \text{ i.e., } Lf'(\xi) \geq 0.$$

Now since $f'(\xi)$ is supposed to exist, $Rf'(\xi) = Lf'(\xi)$ and this is possible only if $f'(\xi) = 0$.

The proof when $f(\xi)$ is a minimum is exactly similar.

9.3. Alternative condition for an extremum.

If $f(x)$ has a continuous second derivative $f''(x)$, we may state the conditions for a maximum and a minimum in another way as follows :

Theorem. If $f'(\xi) = 0$ and $f''(\xi) \neq 0$, then $f(x)$ has an extremum at $x = \xi$, a maximum if $f''(\xi) < 0$ and a minimum if $f''(\xi) > 0$.

Supposing $f''(x) < 0$ in $0 < |x - \xi| < \delta$; then in this neighbourhood of ξ , $f'(x)$ must be a decreasing function and as $f'(\xi) = 0$, it must change from a positive value to a negative value passing through a zero-value at $x = \xi$. Hence $f(x)$ is a maximum at ξ . Moreover as $f''(x)$ is continuous and negative throughout the interval, $f''(\xi) < 0$.

Again if $f''(x) > 0$ in $0 < |x - \xi| < \delta$; then in this interval $f'(x)$ is an increasing function and as $f'(\xi) = 0$, it must increase from a negative value to a positive value passing through a zero-value at ξ . Hence $f(x)$ is a minimum at ξ , $f''(x)$ being positive throughout the interval, $f''(\xi) > 0$.

If $f''(\xi) = 0$, the case is best treated by an application of Taylor's theorem.

9.4. Application of Taylor's theorem in finding extreme values of a function.

In what follows, we shall suppose that $f(x)$ and its derivatives which occur are all continuous.

Theorem. Let $f(x)$ be a function such that $f'(\xi) = f''(\xi) = \dots = f^{(n-1)}(\xi) = 0$, but $f^{(n)}(\xi) \neq 0$. Then $f(\xi)$ is

- (i) a maximum value of $f(x)$ if $f^{(n)}(\xi) < 0$ and n is even ;
- (ii) a minimum value of $f(x)$ if $f^{(n)}(\xi) > 0$ and n is even ;
- (iii) neither a maximum nor a minimum if n is odd.

Expanding $f(\xi+h)$ by Taylor's theorem with remainder after n terms,

we get
$$f(\xi+h) = f(\xi) + hf'(\xi) + \frac{h^2}{2!}f''(\xi) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(\xi) + \frac{h^n}{n!}f^{(n)}(\xi+\theta h), \quad 0 < \theta < 1;$$
 so that by virtue of the conditions stated above

$$f(\xi+h) - f(\xi) = \frac{h^n}{n!}f^{(n)}(\xi+\theta h).$$

By supposition $f^{(n)}(x)$ is continuous and $f^{(n)}(\xi) \neq 0$; we can therefore find a neighbourhood of ξ , viz., $0 < |x - \xi| < \delta$, at every point of which $f^{(n)}(x)$ has the same sign as that of $f^{(n)}(\xi)$. Thus for every point of this neighbourhood $f^{(n)}(\xi+\theta h)$ has the sign of $f^{(n)}(\xi)$.

Let n be even. If $f^{(n)}(\xi) < 0$, $f(\xi+h) - f(\xi)$ has the same sign as that of $\frac{h^n}{n!}f^{(n)}(\xi)$ and whether h be positive or negative, $\frac{h^n}{n!}$ is always positive, so that $f(\xi+h) - f(\xi)$ is always negative whether h be positive or negative. Therefore $f(\xi)$ is a maximum value of $f(x)$. If $f^{(n)}(\xi) > 0$, it can be seen as above that $f(\xi+h) - f(\xi)$ is always positive whether h is positive or negative. So $f(\xi)$ is a minimum value of $f(x)$.

Let n be odd. Whatever definite sign $f^{(n)}(\xi)$ may have, as $\frac{h^n}{n!}$ changes sign with the change in the sign of h , $f(\xi+h) - f(\xi)$ also changes sign with h . Hence $f(\xi)$ is neither a maximum nor a minimum.

9.5. Alternative treatment.

Let $f(x)$ be a function possessing derivatives of the first n orders in the neighbourhood $0 < |x - \xi| < \delta$ of $x = \xi$; suppose further that $f'(\xi) = f''(\xi) = \dots = f^{(n-1)}(\xi) = 0$ and $f^{(n)}(x)$ does not vanish at any point of this neighbourhood.

By Taylor's theorem we evidently have

$$f(\xi+h) - f(\xi) = \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(\xi + \theta h)$$

where $0 < |h| < \delta$ and $0 < \theta < 1$.

Suppose $f^{(n)}(x) < 0$; then $f^{(n-1)}(x)$ is a decreasing function. Therefore for $h > 0$, $f^{(n-1)}(\xi + \theta h) < f^{(n-1)}(\xi) = 0$,

$$\text{i.e., } f^{(n-1)}(\xi + \theta h) < 0.$$

Again for $h < 0$, $f^{(n-1)}(\xi + \theta h) > f^{(n-1)}(\xi) = 0$,

$$\text{i.e., } f^{(n-1)}(\xi + \theta h) > 0.$$

If now n be even,

$$f(\xi+h) - f(\xi) < 0 \text{ whether } h \text{ be } > 0, \text{ or } < 0.$$

$$\therefore f(x) \text{ is a maximum at } x = \xi.$$

If n be odd,

$$f(\xi+h) - f(\xi) < 0 \text{ when } h > 0,$$

$$\text{and } f(\xi+h) - f(\xi) > 0 \text{ when } h < 0.$$

$$\therefore f(\xi) \text{ is not an extremum.}$$

Next let $f^{(n)}(x) > 0$, then $f^{(n-1)}(x)$ is an increasing function.

So that for $h > 0$, $f^{(n-1)}(\xi + \theta h) > f^{(n-1)}(\xi) = 0$,

$$\text{i.e., } f^{(n-1)}(\xi + \theta h) > 0;$$

and for $h < 0$, $f^{(n-1)}(\xi + \theta h) < f^{(n-1)}(\xi) = 0$

$$\text{i.e., } f^{(n-1)}(\xi + \theta h) < 0.$$

If n be even,

$$f(\xi+h) - f(\xi) > 0 \text{ whether } h \text{ be } > 0 \text{ or } < 0.$$

$$\therefore f(x) \text{ is a minimum at } x = \xi.$$

If n be odd,

$$f(\xi+h) - f(\xi) > 0 \text{ when } h > 0,$$

$$\text{and } f(\xi+h) - f(\xi) < 0 \text{ when } h < 0.$$

$$\therefore f(\xi) \text{ is not an extremum.}$$

Hence if $f^{(n)}(\xi)$ be the first derivative which does not vanish, and n is even, $f(\xi)$ is a maximum if $f^{(n)}(\xi) < 0$ and a minimum if $f^{(n)}(\xi) > 0$; if n be odd, $f(\xi)$ is neither a maximum nor a minimum.

9'6. Working Rule : The working rule for finding the extreme values of a function is to find the roots of $f'(x)=0$. If for any of these roots, say ξ , $f^{(n)}(\xi)$ is the first derivative which does not vanish and n is even, $f(\xi)$ is an extremum—a minimum if $f^{(n)}(\xi) > 0$ and a maximum if $f^{(n)}(\xi) < 0$. If n be odd, $f(\xi)$ is not an extremum.

9'7. Case where $f'(x)$ tends to infinity.

We have seen that $f(\xi)$ is an extremum if $f'(x)$ changes sign in passing through ξ . Now a function in general changes its sign in passing through a zero value. A function may however change its sign in passing through an infinite value. Hence we infer that the values of x which satisfy $f'(x)=\infty$ may furnish maximum and minimum values.

An example to illustrate this is the function $f(x)=x^{\frac{2}{3}}$. This has a minimum at $x=0$, furnished by $f'(x)=\infty$.

9'8. Stationary value.

A function $f(x)$ is said to have a *stationary value* for $x=\xi$, if $f'(\xi)=0$; then $f(\xi)$ is a stationary or a turning value of $f(x)$. The name stationary arises from the fact that the rate of change of the function $f(x)$ with respect to x is zero for such a value of x .

It may be noted that a maximum or a minimum value is a stationary value, if $f(x)$ has a point of inflexion, its value thereat is a stationary value.

Note. It is easily seen that

- (a) maximum and minimum values of a function occur alternately ;
- (b) a maximum value is not the absolutely greatest value nor a minimum value the absolutely least value. As a matter of fact, a maximum value may be less than a minimum value.

9'9. Indeterminate Forms : True values.

A function $f(x)$ that is in general well defined for a certain range of values of the variable x , may for a particular value of it, a say, take an indeterminate form (such as $0/0$) that has no meaning. It is possible, however, that $f(x)$ may tend to a definite value A as x tends to a . Although $f(x)$ is really undefined and has no value that can be calculated by the ordinary rules of algebra when $x=a$, yet it is

customary in such a case to call $f(a)$ an *indeterminate form* and to define A as the value of $f(x)$ for $x=a$. The value thus assigned by the definition is called the *critical value* or the *true value* or the *continuity value* of $f(x)$ when $x=a$.

It is to be clearly understood that this 'true value' is assigned by definition and is therefore arbitrary; but there is a certain advantage from the procedure, namely $f(x)$ becomes continuous up to and including the value a , it being supposed that $f(x)$ is in general continuous.

9'10. Typical forms.

A function becomes indeterminate when, for a particular value of the variable, it assumes any of the following forms :

$$0/0, \infty/\infty, \infty - \infty, 0 \times \infty, 0^0, \infty^0, 1^\infty.$$

In many cases the critical value is found most simply by algebraical and trigonometrical transformations and the use of series.

9'11. Method of the Calculus.

I. Form 0/0. L'Hospital's rule.

We get this form when the two terms of the fraction $f(x): F(x)$ which are both continuous vanish simultaneously for $x=a$. The true value is determined by the application of an important rule, known as L'Hospital's rule*, which can be stated in two different forms.

First rule : If the two functions $f(x)$ and $F(x)$ and their successive derivatives up to the $(n-1)$ th order vanish when $x=a$, and if moreover their derivatives of the n th order exist at the point a but are not both zero or both infinite, the true value (finite or infinite) of $f(x): F(x)$ at the point a is $f^{(n)}(a): F^{(n)}(a)$, that is to say

$$\lim_{x \rightarrow a} \frac{f(x)}{F(x)} = \frac{f^{(n)}(a)}{F^{(n)}(a)}.$$

Since the n th derivatives are supposed to exist, if we put $x=a+h$ and expand by Taylor's theorem, we get

$$f(a+h) = \frac{h^n}{n!} M, \quad F(a+h) = \frac{h^n}{n!} M_1,$$

* L'Hospital enunciated this rule only under a geometrical form and that in a very simple case; he borrowed it from Jean Bernoulli.

where M and M_1 have respectively for limits the derivatives $f^{(n)}(a)$ and $F^{(n)}(a)$ when h tends to 0. It follows that

$$\lim_{x \rightarrow a} \frac{f(x)}{F(x)} = \lim_{h \rightarrow 0} \frac{f(a+h)}{F(a+h)} = \lim_{h \rightarrow 0} \frac{M}{M_1} = \frac{f^{(n)}(a)}{F^{(n)}(a)}.$$

Second rule: If $f(x)$ and $F(x)$ vanish at the point a , the true value of $f(x) : F(x)$ at this point is the limit of the quotient $f'(x) : F'(x)$ for $x=a$, provided that this limit, finite or infinite, is determinate. In particular, if $f'(x) : F'(x)$ is still of the form $0/0$, the true value of $f(x) : F(x)$ for $x=a$ is the same as that of $f'(x) : F'(x)$ supposed determinate. This rule holds when we consider only values of $x > a$ or only values $< a$.

The result follows immediately from the mean value theorem of Cauchy. Since $f(a)$ and $F(a)$ both vanish, we get by Cauchy's theorem,

$$\frac{f(a+h)}{F(a+h)} = \frac{f'(a+\theta h)}{F'(a+\theta h)}, \quad (0 < \theta < 1)$$

and it would be sufficient to observe that θh is the same quantity in the two terms of the fraction, that it has the same sign as h and that it tends to 0 with it.

But this rule is subject to the same conditions as the theorem of Cauchy on which it depends, *viz.*, (i) the derivatives should be finite and determinate in the neighbourhood of a (except possibly at a); (ii) as x tends to a , $f'(x) : F'(x)$ may take the form $0/0$ only a limited number of times.

Case where a tends to infinity: If as x increases indefinitely, the derivatives exist and the preceding conditions hold, the second rule is applicable to the case where $a = \infty$ or $-\infty$. Thus when $a = \infty$, we have

$$\lim_{x \rightarrow \infty} \frac{f(x)}{F(x)} = \lim_{x \rightarrow 0} \frac{f\left(\frac{1}{x}\right)}{F\left(\frac{1}{x}\right)} = \lim_{x \rightarrow 0} \frac{-\frac{1}{x^2} f'\left(\frac{1}{x}\right)}{-\frac{1}{x^2} F'\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{f(x)}{F'(x)}.$$

II. **Form $\frac{\infty}{\infty}$.** L'Hospital's second rule is applicable to the determination of the critical values of fractions, the two terms of which increase indefinitely in absolute value for a particular value a of x .

First, let $f(x)$ and $F(x)$ be infinite when x is infinite. Suppose further that $f'(x) : F'(x) \rightarrow A$ when $x \rightarrow \infty$. Let c be a large but finite value of x ; then using Cauchy's theorem and putting x for b and c for a , we have

$$\frac{f(x) - f(c)}{F(x) - F(c)} = \frac{f'(\xi)}{F'(\xi)} \quad (c < \xi < x).$$

Again

$$\frac{f(x) - f(c)}{F(x) - F(c)} = \frac{f(x)}{F(x)} \cdot \frac{1 - f(c)/f(x)}{1 - F(c)/F(x)};$$

hence

$$\frac{f(x)}{F(x)} = \frac{f'(\xi)}{F'(\xi)} \cdot \frac{1 - F(c)/F(x)}{1 - f(c)/f(x)}.$$

Now let c be taken so large that $f'(\xi)/F'(\xi)$ differs from its limit A by less than ϵ_1 and let c be then kept fixed; $f(c)$ and $F(c)$ will, though large, be finite. Then let x be taken so large (and this choice is possible since $f(x)$, $F(x)$ tend to ∞) that the second fraction on the right shall differ from 1 by less than ϵ_2 . The fraction $f(x)/F(x)$ is now the product of two factors, the first of which differs from its limit by less than ϵ_1 and the second of which differs from 1 by less than ϵ_2 , where ϵ_1 and ϵ_2 may be as small as we please. Hence the limit of $f(x)/F(x)$ is A ; that is

$$\lim_{x \rightarrow \infty} \frac{f(x)}{F(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{F'(x)}.$$

Next let $f(a)$ and $F(a)$ be infinite, a being finite. The substitution

$$x = a + \frac{1}{z}$$

reduces the problem to the evaluation of the limit for $z = \infty$, and therefore the theorem holds in this case also.

III. Other forms: The other forms of indetermination can always be reduced to the preceding by suitable transformations. The forms 0° , ∞° , 1^∞ are reduced by taking logarithms.

Examples:

1. If $f(x) = |x|$, show that $f(x)$ is a minimum at $x=0$, although $f'(0)$ does not exist.

2. Determine the extreme values, if any, of

$$\frac{x^4}{(x-1)(x-3)^3},$$

and examine which of them is a maximum and which a minimum.

(C. H. 1964)

3. Determine the finite maximum and minimum values of

$$\frac{(x+1)(x+4)}{(x-1)(x-4)}. \quad (\text{C. H. 1963})$$

4. Discuss the maxima and minima of $x^2/(1-x)^3$. (C. H. 1960)

5. Find the maxima and minima of $\sin x \cos^2 x$ for $0 \leq x \leq 2\pi$.

(C. H. 1964)

6. Find the maximum value $x^m y^n$ if $x+y=k$, a constant, the quantities being all positive. Hence show that

$$a^m b^n < \left(\frac{ma+nb}{m+n} \right)^{m+n}$$

except when $a=b$.

(C. H. 1965)

7. Prove that the maximum of $x^{\frac{1}{x}}$ is at $x=e$. (C. H. 1965)

8. Show that if $a > 1$, the expression $(a-1)a^x - x$ has an extreme value equal to $\log \{(ae-e) \log a\} / \log a$ and determine whether it is a maximum or minimum.

(C. H. 1963)

9. Determine the extreme values of $x^5 - 5x^4 + 5x^3 - 1$.

(C. H. 1965)

10. What is a stationary point of a function? Determine the maximum and minimum values of $12x^5 - 15x^4 - 40x^3 + 1$, and show that there is a stationary point at which the function is neither a maximum nor a minimum.

(C. H. 1962)

11. Find $\lim_{x \rightarrow 0} (\cos mx)^{\frac{n}{x^2}}$.

(C. H. 1966)

12. Evaluate (i) $\lim_{x \rightarrow 0} \left\{ (1+x)x^2 e^{\frac{1}{x^2}} \frac{e^x - e^{-x}}{\log(1+x)} \right\}$;

(ii) $\lim_{x \rightarrow 0} \frac{\log x^2}{\log \cot^2 x}$; (iii) $\lim_{x \rightarrow 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1+x) \right]$;

(iv) $\lim_{x \rightarrow 0} \frac{x - \sin^{-1} x}{\sin^3 x}$; $\lim_{x \rightarrow 0} (\cos x)^{\cot^2 x}$. (C. H. 1965)

13. Find the value of

$$(i) \lim_{x \rightarrow \frac{1}{2}} \frac{\cos^2 \pi x}{e^{\frac{1}{2\pi}} - 2ex}; \quad (ii) \lim_{x \rightarrow 0} (1 + \sin x)^{\cot \delta x},$$

$$(iii) \lim_{x \rightarrow 0} \frac{x^2 + 2 \cos x - 2}{x \sin^3 x}; \quad (iv) \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}.$$

(C. H. 1964)

14. Evaluate (i) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$; (ii) $\lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{\tan^2 \pi x}$;

$$(iii) \lim_{x \rightarrow 1} x^{\tan \frac{\pi x}{2}}; \quad (iv) \lim_{x \rightarrow 0} \frac{e^x - e \sin x}{x - \sin x}. \quad (C. U. 1963)$$

15. Determine the value of

$$(i) \lim_{x \rightarrow 0} (\cos x)^{\cot^2 x}; \quad (ii) \lim_{x \rightarrow 1} \frac{1 - 4 \sin^2 \frac{\pi x}{6}}{1 - x^2}.$$

(C. H. 1961)

16. Show that

$$(i) \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{1}{2}e;$$

$$(ii) \lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e + \frac{1}{2}ex}{x^2} = \frac{11}{24}e.$$

17. If $f(x) = x^2 \sin \frac{1}{x}$ and $\phi(x) = \sin x$, show that although

$$\lim_{x \rightarrow 0} \frac{f'(x)}{\phi'(x)} \text{ does not exist, } \lim_{x \rightarrow 0} \frac{f(x)}{\phi(x)} \text{ exists and is equal to } 0.$$

8. If $f(x) = x - \sin x$ and $\phi(x) = x$, $\lim_{x \rightarrow \infty} \frac{f(x)}{\phi(x)}$ exists and is equal

to 1, but $\lim_{x \rightarrow \infty} \frac{f'(x)}{\phi'(x)}$ does not exist.

19. Prove that

$$(i) \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \{f(x+1) - f(x)\},$$

$$(ii) \lim_{x \rightarrow +\infty} \{f(x)\}^{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{f(x+1)}{f(x)},$$

provided the limits on the right-hand sides exist. (Cauchy)

CHAPTER 10

RIEMANN'S THEORY OF INTEGRATION

10.1. Primitives and Integrals.

Given a function $f(x)$, if there is a function $\phi(x)$ such that $\phi'(x)=f(x)$, $\phi(x)$ is called the **indefinite integral** or **primitive** of $f(x)$. If $\phi(x)$ be a primitive of $f(x)$, $\phi(x)+c$ is also a primitive of $f(x)$, c being an arbitrary constant.

The part of integral calculus which treats of integration as the inverse of differentiation may be called the *calculus of primitives*.

The integral on the other hand, when viewed geometrically, is an area. If $y=f(x)$ has a continuous graph, the area included between the curve representing $f(x)$, the x -axis and the ordinates at $x=a$ and $x=b$ is called a **definite integral** and represented symbolically by

$$\int_a^b f(x) dx.$$

As Analysis is to be independent of geometry, we shall presently give an analytical definition of the integral as the limit of a sum.

The Fundamental Theorem of the integral calculus, which we shall prove later on, *viz.*,

$$\int_a^b f(x) dx = F(b) - F(a),$$

where $F(x)$ is the primitive of $f(x)$, establishes some connection between primitives and integrals; but this does not go far. The distinction between the two is that while integrals can be *calculated*, there are no general rules for the calculation of primitives.

10.2. Analytical Definition of a Definite Integral. Theories of Integration.

There are mainly two theories of integration; one due to Riemann and another, and a more recent one, due to Lebesgue. Whereas the latter theory is more general, the former is much simpler. We shall discuss here Riemann's theory of integration.

Riemann's Theory of Integration.

Let $f(x)$ be a bounded function, continuous or discontinuous, in the interval (a, b) , $a < b$. Divide (a, b) into a finite number n of sub-intervals by taking intermediate points

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Let M and m be the upper and lower bounds of $f(x)$ in (a, b) and M_r and m_r be the corresponding bounds of $f(x)$ in the r th sub-interval (x_{r-1}, x_r) and let δ_r be its length. Let

$$S = (x_1 - a)M_1 + (x_2 - x_1)M_2 + \dots + (b - x_{n-1})M_n = \sum_1^n M_r \delta_r,$$

$$\text{and } s = (x_1 - a)m_1 + (x_2 - x_1)m_2 + \dots + (b - x_{n-1})m_n = \sum_1^n m_r \delta_r.$$

$S = \sum M_r \delta_r$ is called an *upper summation* or an *upper approximate sum* for the integral of $f(x) dx$; and $s = \sum m_r \delta_r$ is called a *lower summation* or a *lower approximate sum* for the same integral.

$$\text{Now } m \leq m_r \leq M_r \leq M;$$

$$\therefore \sum m \delta_r \leq \sum m_r \delta_r \leq \sum M_r \delta_r \leq \sum M \delta_r$$

$$\text{or } m(b-a) \leq s \leq S \leq M(b-a).$$

Hence both S and s are confined between $M(b-a)$ and $m(b-a)$.

It is clear that if we divide the interval (a, b) into sub-intervals in any other way, either by changing the positions of the points $x_1, x_2, \dots, x_{n-1}$ or by altering their number and form corresponding sums S and s , they are also confined between $M(b-a)$ and $m(b-a)$.

Also the sums S and s for any one mode of division of (a, b) must satisfy the relation

$$S \geq s.$$

We can prove also that no S can be less than any s , i.e., $S \geq s$ even when they are formed from two different modes of division.

[Suppose each of the sub-intervals $(a, x_1), (x_1, x_2), \dots, (x_{n-1}, b)$ be divided into smaller intervals and let

$$a, y_1, y_2, \dots, y_{k-1}, x_1, y_{k+1}, \dots, y_{l-1}, x_2, y_{l+1}, \dots, b$$

be the points of division of the new system. This new mode of division is called *consecutive* to the first. Let Σ and σ be the upper and lower sums with respect to this new mode of division of (a, b) .

Let us compare the portions of the two sums Σ and S contributed by the interval (a_1, x_1) . Let $M'_1, M'_2, \dots, M'_k$ be the upper bounds

of $f(x)$ in $(a, y_1), (y_1, y_2), \dots (y_{k-1}, x_1)$. The position of Σ which comes from (a_1, x_1) is

$$M'_1(y_1 - a) + M'_2(y_2 - y_1) + \dots + M'_k(x_1 - y_{k-1}),$$

and since $M'_1, M'_2, \dots, M'_k$ are $\leq M_1$, the above sum is $\leq M_1(x_1 - a)$. Similarly the part of Σ contributed by (x_1, x_2) is $\leq M_2(x_2 - x_1)$ and so on.

$$\therefore \Sigma \leq S.$$

It can be shown in the same way that $\sigma \geq s$.

Let us now consider any two modes of sub-division and S, s and S', s' be the corresponding upper and lower sums. If we suppose the two modes of sub-division to co-exist, we get a third mode which may be considered as consecutive to either of the two.

$$\therefore \Sigma \leq S, \sigma \geq s; \quad \Sigma \leq S', \sigma \geq s'. \quad \text{Also } \Sigma \geq \sigma.$$

$$\text{Hence } s \leq \sigma \leq \Sigma \leq S' \text{ and } s' \leq \sigma \leq \Sigma \leq S, \\ \text{or } S \geq s' \text{ and } S' \geq s.]$$

As S is never less than $m(b-a)$, it has a lower bound I .

Again since s is never greater than $M(b-a)$, it has an upper bound I' . Further as none of the sums S is less than any of the sums s , the bound I cannot be less than I' .

$$\therefore I \geq I'.$$

The lower bound I of the upper sums S is called the *upper integral* of $f(x) dx$ in (a, b) and denoted by

$$I = \int_a^b f(x) dx.$$

The upper bound I' of the lower sums s is called the *lower integral* of $f(x) dx$ in (a, b) and is denoted by

$$I' = \int_a^b f(x) dx$$

The existence of the bounds, I and I' , are assured by the following theorem of Darboux, which we assume without proof.

10.3. Darboux's theorem.

The sums S and s tend respectively to I and I' when the points of division of (a, b) are increased indefinitely so that all the partial intervals tend to zero.

Stated more precisely: If the positive number ϵ is chosen,

however small, there is a positive number δ such that for all modes of division of (a, b) into partial intervals in which all of them are less than or equal to δ , the sum S exceeds I by less than ϵ and the sum s falls short of I' by less than ϵ , that is, $S - I < \epsilon$ and $I' - s < \epsilon$.

[For proof see Carslaw, *Theory of Fourier's Series and Integrals*, Art. 40, p. 79; or Goursat, *Mathematical Analysis*, Art. 72, p. 151.]

10.4. Integrability of a bounded function: Necessary and sufficient condition.

A bounded function $f(x)$ defined in an interval (a, b) is said to be *integrable** in that interval if the two sums S and s approach the same limit when the number of partial intervals is indefinitely increased in such a way that each of the partial intervals approaches zero.

The necessary and sufficient condition that a function be integrable in an interval is that corresponding to any positive number ϵ , another positive number δ exists such that $S - s$ is less than ϵ whenever each of the partial intervals is less than δ .

The condition is *necessary*, for if S and s have the same limit I , we can find a positive number δ so small that $|S - I|$ and $|s - I|$ are each less than $\frac{\epsilon}{2}$ whenever each of the partial intervals is less than δ .

$$\begin{aligned} \text{Now } S - s &= S - I + I - s \\ &\leq |S - I| + |s - I| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

The condition is also *sufficient*.

We are given $S - s < \epsilon$.

$$\text{Now } S \geq I \text{ and } s \leq I'.$$

$$\therefore I - I' \leq S - s < \epsilon.$$

Thus $I = I'$, and the function is integrable.

* That is integrable according to Riemann, which is often denoted as 'R-integrable'.

Thus when the condition is satisfied, the function is integrable in (a, b) and the common limit I is called a *definite integral*, represented

by the symbol $\int_a^b f(x) dx$ and is read "the definite integral of $f(x) dx$ between the limits a and b ". a is called the lower limit and b the upper limit.

10'5. Another definition of a definite integral.

Consider the sum

$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + \dots + f(\xi_r)(x_r - x_{r-1}) + \dots + f(\xi_n)(b - x_{n-1})$, where ξ_r is any value of x in δ_r , i.e., in (x_{r-1}, x_r) . Then since $m_r \leq f(\xi_r) \leq M_r$ where m_r and M_r are the lower and upper bounds in δ_r , $\Sigma m_r \delta_r \leq \Sigma f(\xi_r) \delta_r \leq \Sigma M_r \delta_r$,

$$\text{i.e., } s \leq \Sigma f(\xi_r) \delta_r \leq S, \quad \dots (1)$$

s and S being the lower and upper sums for the above mode of division of (a, b) .

If now $f(x)$ be bounded and integrable in (a, b) ,

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \int_a^b f(x) dx.$$

By Darboux's theorem

$$S < \int_a^b f(x) dx + \epsilon = \int_a^b f(x) dx + \epsilon \quad \dots (2)$$

$$\text{and } s > \int_a^b f(x) dx - \epsilon = \int_a^b f(x) dx - \epsilon \quad \dots (3)$$

where the partial intervals are each less than δ .

From (1), (2), (3), we find that when the partial intervals are each less than δ ,

$$\int_a^b f(x) dx - \epsilon < \sum_1^n f(\xi_r) \delta_r < \int_a^b f(x) dx + \epsilon,$$

$$\text{i.e., } \left| \sum_1^n f(\xi_r) \delta_r - \int_a^b f(x) dx \right| < \epsilon.$$

Observe that when all the partial intervals tend to zero, n tends to infinity, and we get

$$\lim_{n \rightarrow \infty} \sum_{r=1}^n f(\xi_r) \delta_r = \int_a^b f(x) dx.$$

Note. Cauchy's definition of the integral of a *continuous* function $f(x)$ between the limits a and b :

Cauchy took ξ_r to be the value of $f(x)$ at one end of the interval δ_r , so that

$$\int_a^b f(x) dx = \lim \{ (x_1 - a)f(a) + (x_2 - x_1)f(x_1) + \dots + (b - x_{n-1})f(x_{n-1}) \}$$

when the greatest of the sub-intervals (x_{r-1}, x_r) tends to zero. It is obviously a particular case of the above definition.

10.6. Classes of Integrable Functions.

(a) A function continuous in a closed interval is integrable in the interval.

Let $f(x)$ be a function continuous in the closed interval (a, b) ; it is therefore bounded in (a, b) .

A continuous function has the property that corresponding to any positive number ϵ' , there exists a positive number η such that the oscillation of the function is less than ϵ' in any interval of length less than η (Art. 6.5, VI, Cor. II). Dividing (a, b) into a finite number of sub-intervals δ_r , each less than η , the oscillation of $f(x)$ in each sub-interval is less than ϵ' . If ω_r be the oscillation in δ_r , i.e., $\omega_r = M_r - m_r$, M_r and m_r being the upper and lower bounds of $f(x)$ in δ_r , then $\omega_r < \epsilon'$ and we have

$$S - s = \sum (M_r - m_r) \delta_r = \sum \omega_r \delta_r < \epsilon' \sum \delta_r = \epsilon' (b - a),$$

where S and s are the upper and lower sums for the particular mode of division. If we take $\epsilon' = \epsilon / (b - a)$, we get

$$S - s < \epsilon,$$

showing that $f(x)$ is integrable in (a, b) .

(b) A monotone function (increasing or decreasing) is integrable.

Take a monotone increasing function, $f(x)$ in (a, b) . Then $f(x)$ is bounded in (a, b) , $f(a)$ and $f(b)$ being the lower and upper bounds.

Divide (a, b) into n sub-intervals, each less than η , by the points

$$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Then $S = f(x_1)(x_1 - a) + f(x_2)(x_2 - x_1) + \dots + f(b)(b - x_{n-1})$,

$$s = f(a)(x_1 - a) + f(x_1)(x_2 - x_1) + \dots + f(x_{n-1})(b - x_{n-1}).$$

$$S - s = (x_1 - a)\{f(x_1) - f(a)\} + (x_2 - x_1)\{f(x_2) - f(x_1)\} + \dots \\ + (b - x_{n-1})\{f(b) - f(x_{n-1})\}$$

$$< \eta\{f(x_1) - f(a) + f(x_2) - f(x_1) + \dots + f(b) - f(x_{n-1})\}$$

$$< \eta\{f(b) - f(a)\}.$$

The function being bounded, $f(b) - f(a)$ is a definite finite quantity and if we take

$$\eta < \frac{\epsilon}{f(b) - f(a)},$$

we have $S - s < \epsilon$.

Hence $f(x)$ is integrable in (a, b) .

A similar reasoning will hold for a monotone decreasing function.

(c) *A bounded function with an infinite number of finite discontinuities is integrable if the points of discontinuity can be enclosed in intervals the sum of which tends to zero when the intervals are indefinitely diminished.*

Let the bounded function $f(x)$ defined in (a, b) be discontinuous only at the infinite set of points $P_1, P_2, \dots, P_i, \dots$; enclose P_i in an interval Δ_i of which the length is as small as we please; then if $\sum \Delta_i = 0$, the discontinuous function is integrable in (a, b) .

Now $S - s = \sum \omega_i \delta_i$.

The intervals δ_i may be divided into two classes—(1) those which enclose a point of discontinuity of $f(x)$; (2) those which do not. Denoting an interval of the first type by δ_i' and one of the second type by δ_i'' , we may write

$$S - s = \sum \omega_i' \delta_i' + \sum \omega_i'' \delta_i''$$

$$\leq \Omega' \sum \delta_i' + \Omega'' \sum \delta_i'',$$

where Ω' and Ω'' are the greatest of ω_i' and ω_i'' respectively.

Now $\sum \delta_i''$ is a finite quantity, not exceeding $b - a$, equal to k say; again as $\delta_i \rightarrow 0$, we can make δ_i'' as small as we please. Also we can make Ω'' which is the greatest of $\omega_i'' < \frac{\epsilon}{2k}$.

Next $f(x)$ being bounded in (a, b) , the oscillation of $f(x)$ in any sub-interval of (a, b) is finite; thus all ω_i' are finite therefore also Ω' ($=l$ say).

Further, by hypothesis, the points at which $f(x)$ is discontinuous can be enclosed in a set of intervals, the sum of whose lengths can be made as small as we please.

Therefore we can take $\Sigma \delta_i' < \frac{\epsilon}{2l}$.

$$\begin{aligned} \text{Hence } S-s &< \frac{\epsilon}{2l} \cdot l + \frac{\epsilon}{2k} \cdot k \\ &< \epsilon \end{aligned}$$

and so $f(x)$ is integrable.

Cor. 1. As a particular case of the above a bounded function having a finite number of finite discontinuities is integrable.

Cor. 2. If a function $f(x)$ is bounded in (a, b) and the set of its points of discontinuity has only a *finite number of limiting points*, then $f(x)$ is integrable in (a, b) .

Note 1. It is to be remembered that in the definition of the definite integral

$\int_a^b f(x) dx$, the function $f(x)$ is bounded and the limits a and b are finite.

Note 2. When (a, b) is divided into sub-intervals, the sub-interval of greatest length is called the *norm* of the division, and to say that all the sub-intervals tend to zero is the same as to say that the norm of the division tends to zero.

Examples :

1. Show that for the function defined by

$$\begin{aligned} f(x) &= x \sin \frac{1}{x^2} - \frac{1}{x} \cos \frac{1}{x^2} \quad (x \neq 0) \\ &= 0 \quad (x = 0) \end{aligned}$$

there exists a primitive but no integral in $(-1, 1)$.

[$f(x)$ has a primitive $\frac{1}{2}x^2 \sin \frac{1}{x^2}$ for $x \neq 0$ and 0 for $x=0$; but in $(-1, 1)$ it has no integral.]

2. If $f(x)=0$ ($x \neq 0$) and $f(x)=1$ ($x=0$), $f(x)$ has an integral 0, but no primitive.

[We cannot find a function $\phi(x)$ such that $\phi'(x)=0$ when $x \neq 0$ and $\phi'(x)=1$ when $x=0$; hence $f(x)$ has no primitive. $f(x)$ has one

point of discontinuity, namely $x=0$. Neglecting this, we find that the upper bound and lower bound are both 0 elsewhere.

∴ the upper integral $= 0 =$ the lower integral. Hence $f(x)$ has an integral 0.]

3. $f(x)=k$, when x is an integer,
 $=k'$, when x is not an integer.

Show that $f(x)$ is integrable in any range $(0, m)$.

[$f(x)$ is discontinuous at all integral values of x between 0 and m ; and the number of such points is finite. Neglect these. At all other points, $f(x)$ is continuous; hence the integral exists.]

4. $f(x)=k$, when x is rational
 $=k'$, when x is irrational.

Show that $f(x)$ is not integrable in any range (a, b) .

$$\left[\text{If } k > k', \int_a^b f(x) dx = k(b-a), \int_a^b f(x) dx = k'(b-a); \right.$$

the two are not equal.]

5. If $f(x) = \sqrt{1-x^2}$ when x is rational,
 $=1-x$ when x is irrational.

Show that the upper integral in $(0, 1)$ is $\frac{\pi}{4}$ and that the lower integral is $\frac{1}{2}$. (C. H. 1966)

6. A function is defined in $(0, 1)$ as follows :

$$f(x) = \frac{1}{q} \text{ at all rational points } \frac{p}{q} \text{ in its lowest terms,} \\ = 0 \text{ at all irrational points.}$$

Show that $f(x)$ is integrable in $(0, 1)$ and the integral is 0.

[If ϵ be any positive number, however small, $\frac{1}{q} > \frac{\epsilon}{2}$, i.e., $q < \frac{2}{\epsilon} = m$ say. At a rational point $f(x) = \frac{1}{q}$ and q can be given positive integral values 1, 2, 3, ... Now the set of values of q for which $q < m$ is finite and this finite set of integers can be enclosed in intervals the sum of which δ_1 can be made arbitrarily small, i.e., $< \frac{\epsilon}{2}$. The other rational points, infinite in number and for which $q \geq m$, can be enclosed in intervals the sum of which $\delta_2 = 1 - \delta_1 < 1$. Thus there

are two sets of intervals, the first set containing a finite number of rational points enclosed in δ_1 and the second set containing the other rational points between 0 and 1 enclosed in δ_2 . Both sets contain rational as well as irrational points. The maximum value of the oscillation of $f(x)$ in an interval of the first set is 1 and the part of $S-s$ arising from the first set is $< 1 \cdot \delta_1 < \frac{\epsilon}{2}$; the oscillation of $f(x)$ in an interval of the second set is $< \frac{\epsilon}{2}$ and the part of $S-s$ arising from the second set $< \frac{\epsilon}{2} \cdot \delta_2 < \frac{\epsilon}{2} \cdot 1 = \frac{\epsilon}{2}$. Thus for the whole interval $(0, 1)$, $S-s < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $f(x)$ is integrable in $(0, 1)$.

Since any interval of either set contains irrational points, the lower bound of $f(x)$ in any sub-interval is 0 and so the lower integral is 0. As the function is integrable, the upper integral = the lower integral = 0.

$$\text{So } \int_0^1 f(x) dx = 0.]$$

7. Show that every function which is constant is integrable.

[Let $f(x) = k$, a constant. Prove that

$$\int_a^b f(x) dx = \int_a^b f(x) dx = k(b-a).]$$

8. A function $f(x)$ is bounded in (a, b) , show that

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx ; \int_a^b k f(x) dx = k \int_a^b f(x) dx,$$

if k is a positive constant ; and

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx ; \int_a^b k f(x) dx = k \int_a^b f(x) dx,$$

if k is negative constant.

Hence prove that if $f(x)$ is bounded and integrable in (a, b) , so is $k f(x)$ where k is any constant, and that

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx.$$

[If M_r and m_r be the bounds of $f(x)$ in δ_r , kM_r , km_r are the bounds of $kf(x)$ when k is positive and km_r , kM_r are the upper and lower bounds of $k f(x)$, when $k < 0$.]

9. A function $f(x)$ is integrable in (a, b) , show that

$$(i) \quad \text{Lt} \sum_{r=1}^{r=n} hf(a+rh) = \int_a^b f(x) dx,$$

where $h \rightarrow 0$, $n \rightarrow \infty$ and $nh = b - a$.

$$(ii) \quad \text{Lt} \sum_{p=1}^{p=n} f(ar^{p-1})(ar^p - ar^{p-1}) = \int_a^b f(x) dx$$

when $r \rightarrow 1$, $n \rightarrow \infty$, $r^n = \frac{b}{a}$.

10. Apply Ex. 9 (ii) to evaluate $\int_a^b x^m dx$ where m is a positive integer $\neq 1$.

[Divide (a, b) into n sub-intervals by the points $a, ar, ar^2, \dots, ar^{n-1}, ar^n (= b)$. The intervals tend to 0 as $r \rightarrow 1$. x^m being a continuous function the integral exists. Now

$$\begin{aligned} \int_a^b x^m dx &= \text{Lt}_{r \rightarrow 1} [a^m(ar - a) + (ar)^m(ar^2 - ar) + \dots \\ &\quad + (ar^{n-1})^m(ar^n - ar^{n-1})] \\ &= \text{Lt}_{r \rightarrow 1} [a^{m+1}(r-1) + (ar)^{m+1}(r-1) + \dots + (ar^{n-1})^{m+1}(r-1)] \\ &= \text{Lt}_{r \rightarrow 1} [a^{m+1}(r-1)\{1 + r^{m+1} + r^{2(m+1)} + \dots + r^{(n-1)(m+1)}\}] \\ &= \text{Lt}_{r \rightarrow 1} \left[a^{m+1}(r-1) \cdot \frac{r^{n(m+1)} - 1}{r^{m+1} - 1} \right] \end{aligned}$$

$$= \lim_{r \rightarrow 1} \left[\frac{(ar^n)^{m+1} - a^{m+1}}{\frac{r^{m+1} - 1}{r - 1}} \right]$$

$$= \frac{b^{m+1} - a^{m+1}}{m+1}.$$

11. Use the definition of the definite integral to evaluate

$$(i) \int_a^b \sin x \, dx, \quad (ii) \int_a^b \cos x \, dx, \quad (iii) \int_a^b \sec^2 x \, dx.$$

$$[(i) \text{ We have } \int_a^b \sin x \, dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n \mu_r (x_r - x_{r-1})]$$

where $M_r \geq \mu_r \geq m_r$.

Take $\mu_r = \sin \frac{x_r + x_{r-1}}{2}$ and since $(x_r - x_{r-1}) \rightarrow 0$, replace it by $2 \sin \frac{x_r - x_{r-1}}{2}$ which also tends to 0, as $x_r - x_{r-1} \rightarrow 0$. We have thus

$$\int_a^b \sin x \, dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n 2 \sin \frac{x_r + x_{r-1}}{2} \sin \frac{x_r - x_{r-1}}{2}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n (\cos x_{r-1} - \cos x_r)$$

$$= \lim_{n \rightarrow \infty} (\cos x_0 - \cos x_n)$$

$$= \cos a - \cos b.$$

(iii) Take $\mu_r = \sec x_r$, $\sec x_{r-1}$ and replace $(x_r - x_{r-1})$ by $\sin(x_r - x_{r-1})$.]

12. Apply Ex. 9 (ii) to evaluate $\int_a^b \frac{1}{x} \, dx$.

13. Give a geometrical representation of the definite integral

$$\int_a^b f(x) \, dx.$$

14. If $f(x) = \frac{1}{n}$, $\left(\frac{1}{n} \geq x > \frac{1}{n+1}\right)$,

$$f(0) = 0,$$

show that $f(x)$ is integrable in $(0, 1)$ and $\int_0^1 f(x) dx = \frac{\pi^2}{6} - 1$.

[We have $f(x) = 0$ when $x = 0$,

$$f(x) = 1 \quad \text{when } \frac{1}{2} < x \leq 1,$$

$$f(x) = \frac{1}{2} \quad \text{when } \frac{1}{3} < x \leq \frac{1}{2},$$

$$f(x) = \frac{1}{3} \quad \text{when } \frac{1}{4} < x \leq \frac{1}{3},$$

... ..

$$f(x) = \frac{1}{n} \quad \text{when } \frac{1}{n+1} < x \leq \frac{1}{n},$$

... ..

Since $f(x)$ is bounded and monotonically increasing in $(0, 1)$, it is integrable.

Or, Since $f(x)$ is continuous in $(0, 1)$, except at the set of points

$$0, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \dots$$

which has one limiting point, *viz.*, 0, it is integrable. Neglecting these points, the integral is

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[1 \cdot \left(1 - \frac{1}{2}\right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{3} \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \frac{1}{n} \left(\frac{1}{n} - \frac{1}{n+1}\right) \right] \\ &= \lim_{n \rightarrow \infty} \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} - \left(\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}\right) \right] \\ &= \lim_{n \rightarrow \infty} \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} - \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1}\right) \right] \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \text{ to inf. } - \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) \\ &= \frac{\pi^2}{6} - 1. \end{aligned}$$

15. Show that the function defined as follows :

$$f(x) = \frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \quad (n=0, 1, 2, 3, \dots)$$

$$f(0)=0,$$

is integrable in $(0, 1)$ and $\int_0^1 f(x) dx = \frac{2}{3}$.

[$f(x)$ is continuous in $(0, 1)$, except at the set of points

$$0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots$$

which has one limiting point 0, hence $f(x)$ is integrable in $(0, 1)$. Neglecting the points of discontinuity, the integral is,

$$= \text{Lt} \left[1 \left(1 - \frac{1}{2} \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{2^2} \right) + \frac{1}{2^2} \left(\frac{1}{2^2} - \frac{1}{2^3} \right) + \dots + \frac{1}{2^n} \left(\frac{1}{2^n} - \frac{1}{2^{n+1}} \right) + \dots \right]$$

$$= \text{Lt} \left[\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots + \frac{1}{2^{2n+1}} + \dots \right]$$

$$= \frac{1}{2} \cdot \text{Lt} \left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \dots \right]$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{2}{3}]$$

CHAPTER 11

MEAN VALUE & OTHER THEOREMS

11.1. Some properties of the Definite Integral.

$$(a) \quad \int_a^b f(x) dx = - \int_b^a f(x) dx.$$

We have seen (Art. 10.5) that

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n f(\xi_r)(x_r - x_{r-1}).$$

In this it is supposed that $a < b$. If we interchange a and b , that is, if $b < a$, each of the factors $(x_r - x_{r-1})$ changes sign, but $f(\xi_r)$ retains its sign; hence

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

(b) If $a < c < b$,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx. \quad \dots (1)$$

More explicitly :

If a bounded function is integrable in (a, b) , it is also integrable in (a, c) and (c, b) where c is a point of (a, b) . Conversely if $f(x)$ is bounded and integrable in both (a, c) and (c, b) , it is also integrable in (a, b) and in either case, the equality (1) holds.

Let (a, b) be divided in sub-intervals so that c is always a point of division. Let S, S_1, S_2 be the upper sums of $f(x)$ in $(a, b), (a, c)$ and (c, b) respectively for a particular mode of division and s, s_1, s_2 be the lower sums of $f(x)$ in the corresponding intervals. Then

$$S = S_1 + S_2 \quad \text{and} \quad s = s_1 + s_2.$$

$$\therefore S - s = S_1 + S_2 - (s_1 + s_2) = S_1 - s_1 + S_2 - s_2.$$

If $f(x)$ be integrable in (a, b) , $S-s < \epsilon$ when the norm is $< \delta$, and as each of $S-s$, S_1-s_1 , S_2-s_2 is positive,

$$S-s = S_1-s_1 + S_2-s_2 < \epsilon$$

gives $S_1-s_1 < \epsilon$ and $S_2-s_2 < \epsilon$ when the norm is $< \delta$,

$\therefore f(x)$ is integrable in both (a, c) and (c, b) .

Conversely, if $f(x)$ is integrable in both (a, c) and (c, b) , we have,

when the norm is $< \delta$, $S_1-s_1 < \frac{\epsilon}{2}$, $S_2-s_2 < \frac{\epsilon}{2}$.

$$\therefore S-s < \epsilon \text{ (norm } < \delta \text{)}.$$

Thus $f(x)$ is integrable in (a, b) .

$$\text{Let } \int_a^b f(x) dx = \text{Lt } \Sigma f(\xi_r) \delta_r, \quad \int_a^c f(x) dx = \text{Lt } \Sigma f(\xi_p) \delta_p,$$

$$\int_c^b f(x) dx = \text{Lt } \Sigma f(\xi_q) \delta_q.$$

$$\text{Obviously } \Sigma f(\xi_r) \delta_r = \Sigma f(\xi_p) \delta_p + \Sigma f(\xi_q) \delta_q. \quad \dots (2)$$

Also when the norm is $< \delta$,

$$\left| \Sigma f(\xi_r) \delta_r - \int_a^b f(x) dx \right| < \frac{\epsilon}{3}, \quad \left| \Sigma f(\xi_p) \delta_p - \int_a^c f(x) dx \right| < \frac{\epsilon}{3},$$

$$\left| \Sigma f(\xi_q) \delta_q - \int_c^b f(x) dx \right| < \frac{\epsilon}{3}.$$

$$\text{Now } \left| \Sigma f(\xi_r) \delta_r - \int_a^b f(x) dx - \Sigma f(\xi_p) \delta_p + \int_a^c f(x) dx \right.$$

$$\left. - \Sigma f(\xi_q) \delta_q + \int_c^b f(x) dx \right|$$

$$\leq \left| \Sigma f(\xi_r) \delta_r - \int_a^b f(x) dx \right| + \left| \Sigma f(\xi_p) \delta_p - \int_a^c f(x) dx \right|$$

$$+ \left| \sum f(\xi_p) \delta a - \int_a^b f(x) dx \right|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Using (2), we get

$$\left| \int_a^h f(x) dx - \int_a^c f(x) dx - \int_c^b f(x) dx \right| < \epsilon' \quad (\text{norm} < \delta).$$

$$\therefore \int_a^h f(x) dx = \int_a^c f(x) dx + \int_c^h f(x) dx.$$

Cor. If $a < b < c$ and $f(x)$ is bounded and integrable in (a, c) ,

then also $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$

(c) If $f_1(x)$ and $f_2(x)$ be bounded and integrable in (a, b) , so is $f_1(x) + f_2(x)$, and

$$\int_a^b \{f_1(x) + f_2(x)\} dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx.$$

Divide (a, b) into sub-intervals by the points

$$a = x_0 < x_1 < \dots < x_r < \dots < x_{n-1} < x_n = b.$$

Let (M_r', m_r') , (M_r'', m_r'') and (M_r, m_r) be the upper and lower bounds of $f_1(x)$, $f_2(x)$ and $f_1(x) + f_2(x)$ respectively in the r th sub-interval (x_{r-1}, x_r) .

If the upper bounds of $f_1(x)$ and $f_2(x)$ are attained at the same point of the sub-interval, we have $M_r = M_r' + M_r''$; otherwise $M_r < M_r' + M_r''$. Thus

$$M_r \leq M_r' + M_r''.$$

Similarly $m_r \geq m_r' + m_r''.$

If (S, s) , (S_1, s_1) and (S_2, s_2) be the upper and lower sums of

$f_1(x)+f_2(x)$, $f_1(x)$ and $f_2(x)$ respectively for the above mode of division of (a, b) , it follows that

$$S \leq S_1 + S_2,$$

$$s \geq s_1 + s_2.$$

From these we get

$$\int_a^{\bar{b}} \{f_1(x) + f_2(x)\} dx \leq \int_a^{\bar{b}} f_1(x) dx + \int_a^{\bar{b}} f_2(x) dx,$$

$$\text{and } \int_a^{\bar{b}} \{f_1(x) + f_2(x)\} dx \geq \int_a^{\bar{b}} f_1(x) dx + \int_a^{\bar{b}} f_2(x) dx.$$

But as $f_1(x)$ and $f_2(x)$ are integrable in (a, b) , we have

$$\int_a^{\bar{b}} f_1(x) dx = \int_a^b f_1(x) dx; \quad \int_a^{\bar{b}} f_2(x) dx = \int_a^b f_2(x) dx.$$

$$\therefore \int_a^{\bar{b}} \{f_1(x) + f_2(x)\} dx = \int_a^b \{f_1(x) + f_2(x)\} dx.$$

Thus $f_1(x) + f_2(x)$ is integrable in (a, b) and we have

$$\int_a^b \{f_1(x) + f_2(x)\} dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx.$$

Cor. If $f_1(x) > f_2(x)$ in (a, b) ,

$$\int_a^b \{f_1(x) - f_2(x)\} dx = \int_a^b f_1(x) dx - \int_a^b f_2(x) dx.$$

$$\begin{aligned} \left[\int_a^b f_1(x) dx \right] &= \int_a^b \{f_1(x) - f_2(x) + f_2(x)\} dx \\ &= \int_a^b \{f_1(x) - f_2(x)\} dx + \int_a^b f_2(x) dx; \end{aligned}$$

$$\text{hence } \int_a^b f_1(x) dx - \int_a^b f_2(x) dx = \int_a^b \{f_1(x) - f_2(x)\} dx. \quad]$$

(d) If $f(x)$ be integrable in (a, b) , so is $|f(x)|$ and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Since $f(x)$ is integrable in (a, b) , corresponding to any positive number ϵ there exists a positive number δ such that when (a, b) is divided into sub-intervals of norm $< \delta$, we have

$$S - s < \epsilon$$

where S and s are the upper and lower sums for $f(x)$ with respect to the particular mode of division.

If Σ and σ be the corresponding upper and lower sums for $|f(x)|$, using the theorem $||\alpha| - |\beta|| \leq |\alpha - \beta|$, it is easy to see that

$$\Sigma - \sigma \leq S - s.$$

Thus

$$\Sigma - \sigma \leq S - s < \epsilon,$$

showing that $|f(x)|$ is integrable in (a, b) . Again with the usual notation

$$\begin{aligned} & |f(\xi_1)\delta_1 + f(\xi_2)\delta_2 + \dots + f(\xi_n)\delta_n| \\ & \leq |f(\xi_1)|\delta_1 + |f(\xi_2)|\delta_2 + \dots + |f(\xi_n)|\delta_n. \end{aligned}$$

Passing to the limit,

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Note. The converse is however not true. If $|f(x)|$ is integrable $f(x)$ is not necessarily integrable, as the following example will show.

Let $f(x) = 1$ when x is rational,
and $f(x) = -1$ when x is irrational.

then

$$\int_a^b f(x) dx = b - a \quad \text{and} \quad \int_a^b f(x) dx = -(b - a), \quad \text{so that}$$

$$\int_a^b f(x) dx \text{ does not exist. But since } |f(x)| = 1 \text{ for all } x,$$

$$\int_a^b |f(x)| dx \text{ exists and is equal to } (b - a).$$

11.2. First Mean Value theorem for integrals.

Let $f(x)$ and $\phi(x)$ be two bounded integrable functions defined in (a, b) , of which $\phi(x)$ maintains the same sign in (a, b) . Suppose in the first place that $\phi(x) > 0$ and $a < b$.

Let (a, b) be divided into n sub-intervals δ_r ($r=1, 2, 3, \dots, n$) and ξ_r be any point in δ_r . If M and m be the upper and lower bounds of $f(x)$ in (a, b) , then

$$m \leq f(\xi_r) \leq M.$$

Multiply the inequalities by the positive quantities $\phi(\xi_r)\delta_r$. Then adding the inequalities for $r=1, 2, 3, \dots, n$, we get

$$m \sum \phi(\xi_r) \delta_r \leq \sum f(\xi_r) \phi(\xi_r) \delta_r \leq M \sum \phi(\xi_r) \delta_r.$$

If now the number of intervals be increased indefinitely so that each of them tends to zero, we get

$$m \int_a^b \phi(x) dx \leq \int_a^b f(x) \phi(x) dx \leq M \int_a^b \phi(x) dx, \quad \dots \dots (1)$$

which may be written as

$$\int_a^b f(x) \phi(x) dx = \mu \int_a^b \phi(x) dx, \quad \dots \dots (2)$$

where $m \leq \mu \leq M$.

In the case where $\phi(x) < 0$, the signs of the inequalities in (1) will be reversed ; we shall however get the same result as (2).

If in particular $f(x)$ be continuous in (a, b) , $\mu = f(\xi)$ where $a < \xi < b$ and the result (2) takes the form

$$\int_a^b f(x) \phi(x) dx = f(\xi) \int_a^b \phi(x) dx. \quad \dots \dots (3)$$

This is known as the **First Mean Value theorem** for integrals.

Lastly if $\phi(x)=1$, (3) reduces to

$$\int_a^b f(x) dx = (b-a)f(\xi) \quad (a < \xi < b).$$

11.3. The Integral considered as a function of its upper limit : Fundamental Theorem of the Integral Calculus.

Let $f(x)$ be bounded and integrable in (a, b) and let

$$F(x) = \int_a^x f(x) dx,$$

where x is any point of (a, b) . If $(x+h)$ is also in the interval,

$$F(x+h) - F(x) = \int_x^{x+h} f(x) dx = \mu h,$$

(by the Mean Value theorem), where $m \leq \mu \leq M$, M and m being the upper and lower bounds of $f(x)$ in $(x, x+h)$.

It follows that $F(x)$ is a *continuous* function of x in (a, b) .

Further if $f(x)$ is *continuous* in (a, b) ,

$$F(x+h) - F(x) = hf(\xi),$$

where ξ lies between x and $(x+h)$. When $h \rightarrow 0$, $f(\xi) \rightarrow f(x)$.

Therefore

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

Thus when $f(x)$ is continuous in (a, b) , $\int_a^x f(x) dx$ is continuous in (a, b) and has a differential coefficient for every value of x in (a, b) , this differential coefficient being $f(x)$.

This shows that every continuous function is the differential coefficient of another continuous function, usually called its primitive or indefinite integral.

$f(x)$ is continuous in (a, b) , and $F(x) = \int_a^x f(x) dx$, we have seen

just now that

$$\frac{d}{dx} F(x) = f(x).$$

Suppose by some means or other, we obtain a continuous function $\phi(x)$ such that

$$\frac{d}{dx} \phi(x) = f(x).$$

We must then have

$$F(x) = \phi(x) + c$$

where c is an arbitrary constant.

To determine the constant c , we note that $F(x) = 0$ when $x = a$.

$$\therefore 0 = \phi(a) + c, \text{ or } c = -\phi(a);$$

$$\therefore \int_a^x f(x) dx = F(x) = \phi(x) + c = \phi(x) - \phi(a).$$

In particular, when $x = b$,

$$\int_a^b f(x) dx = \phi(b) - \phi(a),$$

where $\phi(x)$ is the primitive of $f(x)$. This result is known as the

Fundamental Theorem of the Integral Calculus.

Note. If $f(x)$ be integrable in (a, b) and $F(x) = \int_a^x f(x) dx$ where x is a point of (a, b) , then

- (1) $F(x)$ is a continuous function of x in (a, b) whether $f(x)$ is so or not ;
- (2) at the point x where $f(x)$ is continuous, $F(x)$ has a derivative, and $F'(x) = f(x)$

(3) if there be a function $\phi(x)$ such that $\phi'(x)=f(x)$ and $f(x)$ is continuous in (a, b) , then

$$F(x) = \int_a^x f(x) dx = \phi(x) - \phi(a).$$

11'4. Abel's lemma.

Let $\epsilon_0, \epsilon_1, \dots, \epsilon_p$ be a set of monotonically decreasing positive numbers and $u_0, u_1, \dots, u_p$ the same number of positive or negative quantities. If A and B be respectively the greatest and least of the sums: $S_0 = u_0, S_1 = u_0 + u_1, \dots, S_p = u_0 + u_1 + \dots + u_p$, then the sum

$$S = \epsilon_0 u_0 + \epsilon_1 u_1 + \dots + \epsilon_p u_p$$

lies between $A\epsilon_0$ and $B\epsilon_0$, that is $A\epsilon_0 \geq S \geq B\epsilon_0$.

For $u_0 = S_0, u_1 = S_1 - S_0, \dots, u_p = S_p - S_{p-1}$.

$$\begin{aligned} \therefore S &= \epsilon_0 u_0 + \epsilon_1 (S_1 - S_0) + \dots + \epsilon_p (S_p - S_{p-1}) \\ &= S_0(\epsilon_0 - \epsilon_1) + S_1(\epsilon_1 - \epsilon_2) + \dots + S_{p-1}(\epsilon_{p-1} - \epsilon_p) + S_p \epsilon_p. \end{aligned}$$

Since none of the differences $\epsilon_0 - \epsilon_1, \epsilon_1 - \epsilon_2, \dots, \epsilon_{p-1} - \epsilon_p$ is negative, two limits for S are obtained by replacing $S_0, S_1, \dots, S_p$ by their upper limit A and then by their lower limit B . Thus

$$S \leq A(\epsilon_0 - \epsilon_1 + \epsilon_1 - \epsilon_2 + \dots + \epsilon_{p-1} - \epsilon_p + \epsilon_p) \leq A\epsilon_0,$$

and similarly $S \geq B\epsilon_0$.

$$\therefore A\epsilon_0 \geq S \geq B\epsilon_0.$$

11'5. Second Mean Value theorem for integrals. (Bonnet)

If $f(x)$ and $\phi(x)$ be two bounded integrable functions, of which $\phi(x)$ is positive and monotone decreasing in (a, b) , then

$$\int_a^b f(x)\phi(x) dx = \phi(a) \int_a^b f(x) dx \quad (a \leq \xi \leq b).$$

The integral $\int_a^b f(x)\phi(x) dx$ is the limit of the sum

$$\begin{aligned} &f(a)\phi(a)(x_1 - a) + f(x_1)\phi(x_1)(x_2 - x_1) + \dots \\ &\quad + f(x_{i-1})\phi(x_{i-1})(x_i - x_{i-1}) + \dots \end{aligned}$$

as the greatest of the intervals (x_{i-1}, x_i) tends to zero.

The numbers $\phi(a), \phi(x_1), \dots$ form a set of monotonically decreasing positive quantities; hence the above sum, by Abel's lemma, lies

between $A\phi(a)$ and $B\phi(a)$ where A and B are respectively the greatest and least of the sums

$$\begin{aligned} & f(a)(x_1 - a), \\ & f(a)(x_1 - a) + f(x_1)(x_2 - x_1), \\ & \dots \quad \dots \quad \dots \quad \dots \\ & f(a)(x_1 - a) + f(x_1)(x_2 - x_1) + \dots + f(x_{n-1})(b - x_{n-1}). \end{aligned}$$

Passing to the limit, it is clear that the integral in question must lie between $A_1\phi(a)$ and $B_1\phi(a)$ where A_1 and B_1 denote the maximum

and minimum respectively of the integral $\int_a^c f(x) dx$ as c varies from

a to b . For a bounded integrable function $f(x)$, $\int_a^c f(x) dx$ is a conti-

nuous function of its upper limit c ; hence we may write

$$\int_a^b f(x)\phi(x) dx = \phi(a) \int_a^\xi f(x) dx, \quad (a \leq \xi \leq b).$$

Cor. If $\phi(x)$ be positive but *monotone increasing*, the Second Mean Value theorem takes the form

$$\int_a^b f(x)\phi(x) dx = \phi(b) \int_\xi^b f(x) dx, \quad (a \leq \xi \leq b).$$

11'6. Second Mean Value theorem (Weierstrass's form)

If the function $\phi(x)$ is monotone decreasing without being always positive, we may write $\psi(x) = \phi(x) - \phi(b)$; then $\psi(x)$ is monotone decreasing and positive. Hence using Bonnet's formula,

$$\int_a^b f(x)\psi(x) dx = \psi(a) \int_a^\xi f(x) dx, \quad (a \leq \xi \leq b),$$

$$\text{or, } \int_a^b f(x)\{\phi(x) - \phi(b)\} dx = \{\phi(a) - \phi(b)\} \int_a^\xi f(x) dx,$$

$$\begin{aligned}\text{or, } \int_a^b f(x)\phi(x) dx &= \phi(a) \int_a^\xi f(x) dx + \phi(b) \int_a^b f(x) dx - \phi(b) \int_a^\xi f(x) dx, \\ &= \phi(a) \int_a^\xi f(x) dx + \phi(b) \int_\xi^b f(x) dx, \quad (a \leq \xi \leq b).\end{aligned}$$

11.7. Second Mean Value theorem (Alternative treatment).

If $\phi(x)$ be a function which has a derivative of constant sign for all values of x from a to b and $f(x)$ is continuous and so integrable in (a, b) , then there is a number ξ between a and b such that

$$\int_a^b f(x)\phi(x) dx = \phi(a) \int_a^\xi f(x) dx + \phi(b) \int_\xi^b f(x) dx.$$

Since $f(x)$ is continuous and so integrable in (a, b) , let

$$\psi(x) = \int_a^x f(t) dt$$

so that $\psi(x)$ is continuous in (a, b) and as $f(x)$ is continuous in (a, b) , $\psi'(x) = f(x)$ exists at every point of (a, b) . Also

$$\psi(a) = \int_a^a f(t) dt = 0.$$

Now $\int_a^b f(x)\phi(x) dx = \int_a^b \phi(x)\psi'(x) dx$ and as $f(x)\phi(x)$ is continuous,

the integral exists. By integration by parts, it is equal to

$$\left[\phi(x)\psi(x) \right]_a^b - \int_a^b \phi'(x)\psi(x) dx$$

$$= \phi(b)\psi(b) - \int_a^b \phi'(x)\psi(x) dx.$$

Since $\psi(x)$ is continuous and $\phi'(x)$ maintains a constant sign, by the First Mean Value theorem

$$\begin{aligned}\int_a^b \phi'(x)\psi(x) dx &= \psi(\xi) \int_a^b \phi'(x) dx, \quad (a \leq \xi \leq b), \\ &= \psi(\xi)[\phi(b) - \phi(a)]. \\ \therefore \int_a^b f(x)\phi(x) dx &= \phi(b)\psi(b) - \psi(\xi)[\phi(b) - \phi(a)] \\ &= \phi(a)\psi(\xi) + \phi(b)[\psi(b) - \psi(\xi)] \\ &= \phi(a) \int_a^\xi f(x) dx + \phi(b) \int_\xi^b f(x) dx, \\ &\quad (a \leq \xi \leq b) \dots (1)\end{aligned}$$

Note. Since $\phi(x)$ has derivative of a constant sign, $\phi(x)$ is monotone—decreasing if the sign be negative and increasing if it be positive.

Now (1) may be written

$$\begin{aligned}\int_a^b f(x)\phi(x) dx &= \phi(a) \int_a^\xi f(x) dx + \phi(b) \left[\int_a^b f(x) dx - \int_a^\xi f(x) dx \right] \\ \text{or, } \int_a^b f(x)[\phi(x) - \phi'(b)] dx &= [\phi(a) - \phi(b)] \int_a^\xi f(x) dx.\end{aligned}$$

If $\phi'(x) < 0$, $\phi(x)$ is monotone decreasing; if it is not always positive and we put $F(x) = \phi'(x) - \phi(b)$, $F(x)$ is monotone decreasing but always positive; and we have

$$\int_a^b f(x) F(x) dx = F(a) \int_a^\xi f(x) dx, \quad a \leq \xi \leq b \quad \dots (2)$$

Again (1) may be written

$$\begin{aligned}\int_a^b f(x)\phi(x) dx &= \phi(a) \left[\int_a^b f(x) dx - \int_\xi^b f(x) dx \right] + \phi(b) \int_\xi^b f(x) dx, \\ \text{or, } \int_a^b f(x)[\phi(x) - \phi(a)] dx &= [\phi(b) - \phi(a)] \int_\xi^b f(x) dx.\end{aligned}$$

If $\phi'(x) > 0$, $\phi(x)$ is monotone increasing; if it is *not* always positive, on setting $G(x) = \phi(x) - \phi(a)$, $G(x)$ is monotone increasing but always positive, and we have

$$\int_a^b f(x) G(x) dx = G(b) \int_a^b f(x) dx, (a \leq \xi \leq b.) \quad \dots (3)$$

(2) and (3) are Bonnet's form whereas (1) is Weierstrasse's form of the Second Mean Value theorem.

11.8. To establish Taylor's theorem by integration by parts.

To establish Taylor's theorem by integration by parts and to apply the First Mean Value theorem to obtain different forms of the remainder after n terms.

We have by the Fundamental theorem

$$\int_a^{a+h} f'(t) dt = f(a+h) - f(a).$$

Changing the integral on the left by the substitution $t = a + h - x$, we get

$$f(a+h) - f(a) = \int_0^h f'(a+h-x) dx.$$

Integrating by parts in succession, we get, step by step,

$$\begin{aligned} f(a+h) - f(a) &= \int_0^h f'(a+h-x) dx \\ &= hf'(a) + \int_0^h x f''(a+h-x) dx, \\ &= hf'(a) + \frac{h^2}{2!} f''(a) + \frac{1}{2!} \int_0^h x^2 f'''(a+h-x) dx, \\ &\dots \dots \dots \\ &= hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) \\ &\quad + \frac{1}{(n-1)!} \int_0^h x^{n-1} f^{(n)}(a+h-x) dx. \end{aligned}$$

$$\text{Thus } R_n = \frac{1}{(n-1)!} \int_0^h x^{n-1} f^{(n)}(a+h-x) dx.$$

$$\text{Putting } x^{n-1} f^{(n)}(a+h-x) = \phi(x) \psi(x),$$

$$\text{we get } R_n = \frac{1}{(n-1)!} \int_0^h \phi(x) \psi(x) dx = \frac{\psi(\xi)}{(n-1)!} \int_0^h \phi(x) dx, \quad (0 < \xi < h).$$

$$(i) \text{ Taking } \phi(x) = x^{n-1} \text{ and } \psi(x) = f^{(n)}(a+h-x),$$

$$R_n = \frac{f^{(n)}(a+h-\xi)}{(n-1)!} \int_0^h x^{n-1} dx.$$

$$\text{Let } \xi = \theta' h, \quad (0 < \theta' < 1), \text{ then } a+h-\xi = a+h(1-\theta') = a+\theta h \\ (0 < \theta < 1),$$

$$\text{and } R_n = \frac{f^{(n)}(a+\theta h)}{(n-1)!} \cdot \frac{h^n}{n} = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

—Lagrange's form.

(ii) Put $\phi(x) = 1$, and $\psi(x) = x^{n-1} f^{(n)}(a+h-x)$. If $0 < \xi < h$, we can write $\xi = (1-\theta)h$, $(0 < \theta < 1)$.

$$R_n = \frac{\psi(\xi)}{(n-1)!} \int_0^h \phi(x) dx = \frac{(1-\theta)^{n-1} h^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) \cdot \int_0^h dx \\ = \frac{(1-\theta)^{n-1} h^n}{(n-1)!} f^{(n)}(a+\theta h) \quad \text{—Cauchy's form.}$$

$$(iii) \text{ Lastly put } \phi(x) = x^{p-1}, \psi(x) = x^{n-p} f^{(n)}(a+h-x).$$

$$\text{Then } R_n = \frac{\xi^{n-p} f^{(n)}(a+h-\xi)}{(n-1)!} \int_0^h x^{p-1} dx \\ = \frac{(1-\theta)^{n-p} h^n}{(n-1)! p} f^{(n)}(a+\theta h). \quad \text{—Schlömlich's form.}$$

Note. In order that $\int_0^h f^{(n)}(a+h-x) dx$ may exist, we have to assume that

$f^{(n)}(x)$ is continuous; but this assumption is not necessary to establish Taylor's theorem as the generalised mean value theorem.

11.9. Young's form of Taylor's theorem.

If $f^{(n)}(a)$ exists finitely, then

$$\begin{aligned} f(a+h) = & f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots \\ & + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} \{f^{(n)}(a) + \epsilon\} \end{aligned}$$

where $\epsilon \rightarrow 0$ as $h \rightarrow 0$.

$$\text{Let } \psi(h) = \frac{h^n}{n!} \text{ and } \phi(h) = f(a+h) - f(a) - hf'(a) - \dots - \frac{h^n}{n!} f^{(n)}(a).$$

It is easy to see that

$$\phi(0) = \phi'(0) = \dots = \phi^{(n)}(0) = 0,$$

$$\text{and } \psi(0) = \psi'(0) = \dots = \psi^{(n-1)}(0) = 0; \psi^{(n)}(0) = 1.$$

By the theory of indeterminate forms,

$$\lim_{h \rightarrow 0} \frac{\phi(h)}{\psi(h)} = \frac{\phi^{(n)}(0)}{\psi^{(n)}(0)} = \frac{0}{1} = 0;$$

$$\therefore \frac{\phi(h)}{\psi(h)} = \epsilon \text{ where } \epsilon \rightarrow 0 \text{ as } h \rightarrow 0.$$

$$\text{This gives } \phi(h) = \epsilon \psi(h) = \frac{h^n}{n!} \epsilon,$$

$$\begin{aligned} \text{or, } f(a+h) = & f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots \\ & + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} \{f^{(n)}(a) + \epsilon\}. \end{aligned}$$

Examples :

1. If α and ϕ are positive acute angles, prove that

$$\phi < \int_0^\phi \frac{dx}{\sqrt{(1 - \sin^2 \alpha \sin^2 x)}} < \frac{\phi}{\sqrt{(1 - \sin^2 \alpha \sin^2 \phi)}}.$$

If $\alpha = \phi = \frac{\pi}{6}$, then the integral lies between '523 and '541.

2. Prove that

$$\frac{\pi}{4} < \int_0^{\frac{\pi}{4}} \sec x \, dx < \frac{\pi}{2\sqrt{2}}.$$

3. Prove that

$$\int_0^{\pi} \frac{x^2 \, dx}{5 + 3 \cos x} \text{ lies between } \frac{\pi^3}{24} \text{ and } \frac{\pi^3}{6}.$$

4. Use the first mean value theorem for integrals to show that

$$\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}} \leq \frac{\pi}{6} \frac{1}{\sqrt{(1-\frac{1}{4}k^2)}}.$$

5. Prove that

$$\frac{1}{2} \left(1 - \frac{1}{e}\right) < \int_0^1 e^{-x^2} \, dx < 1 \text{ and } 0 < \int_1^{\infty} e^{-x^2} \, dx < \frac{1}{2e},$$

and deduce that

$$\frac{1}{2} \left(1 - \frac{1}{e}\right) < \int_0^{\infty} e^{-x^2} \, dx < 1 + \frac{1}{2e}.$$

[In $(0, 1)$, $x < 1$. $\therefore x e^{-x^2} < e^{-x^2}$. Hence

$$\frac{1}{2} \int_0^1 2x e^{-x^2} \, dx < \int_0^1 e^{-x^2} \, dx;$$

$$\text{or, } \frac{1}{2} \left| -e^{-x^2} \right|_0^1 < \int_0^1 e^{-x^2} \, dx, \text{ i.e., } \frac{1}{2} \left(1 - \frac{1}{e}\right) < \int_0^1 e^{-x^2} \, dx.$$

Now e^{-x^2} is a decreasing function and for $x=0$, $e^{-x^2}=1$.

$$\therefore e^{-x^2} < 1.$$

$$\therefore \int_0^1 e^{-x^2} \, dx < \int_0^1 dx = 1.$$

Combining, we get

$$\frac{1}{2} \left(1 - \frac{1}{e}\right) < \int_0^1 e^{-x^2} \, dx < 1. \quad \dots (1)$$

In $(1, \infty)$, $x > 1$. $\therefore e^{-x^2} < x e^{-x^2}$.

$$\text{Hence } \int_1^{\infty} e^{-x^2} dx < \frac{1}{2} \int_1^{\infty} e^{-x^2} 2x dx = \frac{1}{2} \left[-e^{-x^2} \right]_1^{\infty} = \frac{1}{2e}.$$

Also e^{-x^2} being greater than 0, $\int_1^{\infty} e^{-x^2} dx > 0$.

Combining, we have

$$0 < \int_1^{\infty} e^{-x^2} dx < \frac{1}{2e}. \quad \dots \quad (2)$$

Adding (1) and (2)

$$\frac{1}{2} \left(1 - \frac{1}{e} \right) < \int_0^{\infty} e^{-x^2} dx < 1 + \frac{1}{2e}. \quad]$$

6. If $f(x) \leq \phi(x)$ when $a \leq x \leq b$, prove that

$$\int_a^b f(x) dx \leq \int_a^b \phi(x) dx.$$

7. If $n > 1$ then $\cdot 5 < \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^{2n}}} < \cdot 521$. (C. H. 1963, 1966)

[The first inequality follows from the fact that $\sqrt{1-x^{2n}} < 1$, the second from the fact $\sqrt{1-x^{2n}} > \sqrt{1-x^2}$.]

8. Prove that

$$\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}. \quad (\text{C. H. 1954, '55, '59, '62})$$

9. Prove that

$$\frac{1}{4} \leq \int_0^1 \frac{dx}{4-x^2+x^3} \leq \frac{\pi}{6} \quad (\text{C. H. 1957})$$

10. Show that

$$\cdot 573 < \int_1^2 \frac{dx}{\sqrt{4-3x+x^3}} < \cdot 595.$$

[Put $x=1+u$; then replace $2+3u^2+u^3$ successively by $2+4u^2$ and by $2+3u^2$; here $0 < u < 1$.]

11. If $0 < a < b$, prove that

$$\left| \int_a^b \frac{\sin x}{x} dx \right| < \frac{2}{a} \quad (C. H. 1964, '66)$$

[Apply Bonnet's form of the second mean value theorem and note that the integral of $\sin x$ over any interval is numerically less than 2.]

12. Does Bonnet's theorem hold for the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x dx ?$$

Give reasons for your answer.

(C. H. 1966)

13. Show that as $n \rightarrow \infty$,

$$(i) \quad \frac{1}{\sqrt{(2n-1^2)}} + \frac{1}{\sqrt{(4n-2^2)}} + \frac{1}{\sqrt{(6n-3^2)}} + \dots + \frac{1}{n} = \frac{\pi}{2} ;$$

$$(ii) \quad \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n} = \log 3.$$

14. Find the limit as $n \rightarrow \infty$ of

$$\frac{(n!)^{\frac{1}{n}}}{n}. \quad (C. H. 1964)$$

15. Prove Schwarz's inequality for integrals, viz.,

$$\left(\int_a^b \phi(x)\psi(x) dx \right)^2 \leq \int_a^b \{\phi(x)\}^2 dx \int_a^b \{\psi(x)\}^2 dx.$$

[From Schwarz's inequality (often called Cauchy's inequality), we have $\{\sum \phi(\xi_v) \psi(\xi_v) \delta_v\}^2 \leq \sum \{\phi(\xi_v)\}^2 \delta_v \cdot \sum \{\psi(\xi_v)\}^2 \delta_v$; on proceeding to the limit when the greatest of $\delta_v \rightarrow 0$, we get the required result.]

16. Defining $\log x$ by the definite integral

$$\int_1^x \frac{1}{t} dt \quad (x > 0), \text{ deduce that}$$

$$(i) \log(mn) = \log m + \log n.$$

(C. H. 1966)

$$(ii) \log \left(\frac{m}{n} \right) = \log m - \log n.$$

$$\int (i) \text{In } \log x = \int_1^x \frac{1}{t} dt, \text{ put } t = \frac{1}{u}; \text{ we get}$$

$$\log x = \int_1^x \frac{1}{t} dt = - \int_1^{\frac{1}{x}} \frac{du}{u} = -\log \frac{1}{x}. \quad \dots (1)$$

$$\text{In } \log (mn) = \int_1^{mn} \frac{dt}{t}, \text{ put } t = nu \text{ where } n \text{ is a positive constant};$$

$$\text{then } \log (mn) = \int_1^{mn} \frac{dt}{t} = \int_{\frac{1}{n}}^m \frac{du}{u} = \int_1^m \frac{du}{u} - \int_1^{\frac{1}{n}} \frac{du}{u} = \log m - \log \frac{1}{n}$$

$$= \log m + \log n, \text{ by ().}$$

$$(ii) \text{In } \log \left(\frac{m}{n} \right) = \int_1^{\frac{m}{n}} \frac{dt}{t}, \text{ put } t = \frac{u}{n}; \text{ then}$$

$$\int_1^{\frac{m}{n}} \frac{dt}{t} = \int_n^m \frac{du}{u} = \int_1^m \frac{du}{u} - \int_1^n \frac{du}{u}$$

$$= \log m - \log n.]$$

17. Using the definition : $\log x = \int_1^x \frac{dt}{t}$ ($x > 0$), establish the following results :

$$(a) \frac{x}{1+x} < \log (1+x) < x \quad (x > 0);$$

$$(b) x < -\log (1-x) < \frac{x}{1-x} \quad (0 < x < 1). \quad (O. H. 1963)$$

$$[(a) \log (1+x) = \int_1^{1+x} \frac{dt}{t}. \text{ Here } t \text{ lies between } 1 \text{ and } 1+x,$$

i.e., $1 < t < 1+x$. $\therefore \frac{1}{1+x} < \frac{1}{t} < 1$. From these we get

$$\frac{1}{1+x} \int_1^{1+x} dt < \int_1^{1+x} \frac{dt}{t} < \int_1^{1+x} dt,$$

$$\text{or, } \frac{x}{1+x} < \log(1+x) < x.$$

$$(b) -\log(1-x) = -\int_1^{1-x} \frac{dt}{t} = \int_{1-x}^1 \frac{dt}{t}$$

Here $1-x < t < 1$, or $\frac{1}{1-x} > \frac{1}{t} > 1$.

$$\therefore \int_{1-x}^1 dt < \int_{1-x}^1 \frac{dt}{t} < \frac{1}{1-x} \int_{1-x}^1 dt$$

or, $x < -\log(1-x) < \frac{x}{1-x}$ ($0 < x < 1$). See example 3, Ch.8.)]

8. Defining the number e by the relation $\log e = 1$, that is,

$$\int_1^e \frac{dt}{t} = 1,$$

show that $2 < e < 3$.

$$\left[\text{In } \int_1^2 \frac{dt}{t}, 1 < t < 2 \text{ or } \frac{1}{2} < \frac{1}{t} < 1; \right.$$

$$\therefore \int_1^2 \frac{dt}{t} < \int_1^2 1 \cdot dt = 1 = \int_1^e \frac{dt}{t}; \text{ whence } 2 < e.$$

$$\text{Again } \int_1^3 \frac{dt}{t} = \int_1^2 \frac{dt}{t} + \int_2^3 \frac{dt}{t} = \int_0^1 \frac{du}{2-u} + \int_0^1 \frac{du}{2+u} = 4 \int_0^1 \frac{du}{4-u^2} > 4 \int_0^1 \frac{du}{4}$$

$$= 1 = \int_1^e \frac{dt}{t}; \text{ whence } 3 > e. \quad]$$

CHAPTER 12

IMPROPER INTEGRALS

12.1 Improper Integrals.

The definition of the integral $\int_a^b f(x) dx$ (Chapter 10) requires that

$f(x)$ is bounded in $a \leq x \leq b$ and the range of integration is finite. The definition does not apply to the case where $f(x)$ is not bounded or a limit of integration is infinite. It is possible however to extend

the definition to cover such cases so that $\int_a^b f(x) dx$ in these exceptional cases may sometimes have a definite meaning. Such integrals are called *improper* or *infinite* or *generalised* integrals. An integral which is not improper will often be called a *proper* integral.

12.2 Improper Integrals : Class I.

We first consider improper integrals in which a or b or both become infinite.

(a) When the upper limit b is infinite, we define $\int_a^\infty f(x) dx$ as

$\text{Lt}_{B \rightarrow \infty} \int_a^B f(x) dx$ if the limit exists and $f(x)$ is bounded and integrable in (a, B) .

(b) When the lower limit a tends to $-\infty$, we define

$\int_{-\infty}^b f(x) dx$ as $\text{Lt}_{A \rightarrow -\infty} \int_A^b f(x) dx$ if the limit exists and $f(x)$ is bounded and integrable in (A, b) .

(c) When $a \rightarrow -\infty$ and $b \rightarrow \infty$, we define $\int_{-\infty}^{\infty} f(x) dx$ as the

sum of the limits $A \rightarrow -\infty \int_A^c f(x) dx$, $B \rightarrow \infty \int_c^B f(x) dx$,

$$\begin{aligned} \text{i.e., } \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx \\ &= \lim_{A \rightarrow -\infty} \int_A^c f(x) dx + \lim_{B \rightarrow \infty} \int_c^B f(x) dx, \end{aligned}$$

if both limits exist and $f(x)$ is bounded and integrable in both (A, c) and (c, B) , c being any number chosen conveniently.

If the limit exists in either of (a) or (b) the corresponding improper integral is said to *converge* or to be *convergent*. It is said to *diverge* or to be *divergent* if it tends to infinity with a fixed sign. When it does not converge or diverge, it is said to *oscillate* or to be *oscillatory*. In (c) both limits must exist in order that the integral may converge.

Illustrative Examples :

$$\text{Ex. 1. } \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}; \int_0^{\infty} e^{-x} dx = 1; \int_1^{\infty} \frac{dx}{x^2} = 1; \int_1^{\infty} \frac{dx}{x^3} = 2.$$

$$\text{Ex. 2. } \int_0^{\infty} e^x dx = \infty; \int_1^{\infty} \log \frac{1}{x} dx = -\infty; \int_1^{\infty} \frac{dx}{\sqrt{x}} = \infty; \int_2^{\infty} \frac{dx}{1-x} = -\infty.$$

$$\text{Ex. 3. } \int_a^{\infty} \sin x dx \text{ oscillate finitely; } \int_a^{\infty} x \sin x dx \text{ oscillates}$$

infinitely.

$$\text{Ex. 4. } \int_{-\infty}^{\infty} x e^{-x^2} dx = 0.$$

12'3. Improper Integrals : Class II.

We next consider the following cases of improper integrals in each of which $f(x)$ has an infinite discontinuity.

(a) $f(x)$ has an infinite discontinuity at c where $a < c < b$.

By $\int_a^b f(x) dx$ we shall mean the sum of the two limits :

$$\text{Lt}_{\epsilon \rightarrow 0} \int_a^{c-\epsilon} f(x) dx \text{ and } \text{Lt}_{\delta \rightarrow 0} \int_{c+\delta}^b f(x) dx,$$

$$\text{that is } \int_a^b f(x) dx = \text{Lt}_{\epsilon \rightarrow 0} \int_a^{c-\epsilon} f(x) dx + \text{Lt}_{\delta \rightarrow 0} \int_{c+\delta}^b f(x) dx.$$

If either of these limits does not exist, the integral does not exist. It sometimes happens that when we make $\epsilon = \delta$, both the

limits $\text{Lt}_{\epsilon \rightarrow 0} \int_a^{a-\epsilon} f(x) dx$ and $\text{Lt}_{\epsilon \rightarrow 0} \int_{c+\epsilon}^b f(x) dx$ exist ; $\int_a^b f(x) dx$ has

then a value called by Cauchy the *principal value* of the improper

integral and if $\int_a^b f(x) dx$ when $\epsilon \neq \delta$ exists, it is called the *general*

value of the integral. There may be cases where the principal value exists but not the general value.

(b) If $f(x)$ has an infinite discontinuity at the lower limit a , $\int_a^b f(x) dx$ is then defined by $\text{Lt}_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f(x) dx$, if it exists.

(c) If $f(x)$ has an infinite discontinuity at the upper limit b , by $\int_a^b f(x) dx$ we shall mean $\text{Lt}_{\epsilon \rightarrow 0} \int_a^{b-\epsilon} f(x) dx$, if the limit exists.

In the above cases when the appropriate limits exist, the corresponding improper integral is said to be convergent ; otherwise it is *non-convergent* or *non-existent*.

Illustrative examples :

Ex. 1. $\int_0^1 \frac{1}{x} dx$ does not exist.

Ex. 2. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ converges to $\frac{\pi}{2}$.

Ex. 3. $\int_{-1}^1 \frac{dx}{x^3}$ has a principal value = 0, but the general value does not exist.

12'4. Three important improper integrals.

(a) $\int_a^\infty \frac{dx}{x^\mu}$ ($a > 0$) exists if $\mu > 1$ and does not exist if $\mu \leq 1$.

If $\mu \neq 1$, we have

$$\int_a^X \frac{dx}{x^\mu} = \frac{1}{1-\mu} \left| \frac{1}{x^{\mu-1}} \right|_a^X = \frac{1}{1-\mu} \left[\frac{1}{X^{\mu-1}} - \frac{1}{a^{\mu-1}} \right].$$

$$\therefore \lim_{X \rightarrow \infty} \int_a^X \frac{dx}{x^\mu} = \frac{1}{1-\mu} \lim_{X \rightarrow \infty} \left[\frac{1}{X^{\mu-1}} - \frac{1}{a^{\mu-1}} \right] \text{ which}$$

tends to $\frac{1}{(\mu-1)a^{\mu-1}}$ or ∞ according as $\mu > 1$ or $\mu < 1$.

If $\mu = 1$, $\int_a^X \frac{dx}{x} = \log X - \log a$.

$$\therefore \lim_{X \rightarrow \infty} \int_a^X \frac{dx}{x} = \lim_{X \rightarrow \infty} (\log X - \log a) \text{ which tends to } \infty \text{ as}$$

$X \rightarrow \infty$.

Hence the result.

(b) $\int_a^b \frac{dx}{(x-a)^\mu}$ exists if $\mu < 1$ and does not exist if $\mu \geq 1$.

(When $\mu \leq 0$, the integral is proper.)

$\int_a^b \frac{dx}{(x-a)^\mu}$ has an infinite discontinuity at the lower limit a .

$$\begin{aligned} \text{If } \mu \neq 1, \quad \int_a^b \frac{dx}{(x-a)^\mu} &= \lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)^\mu} \\ &= \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{1-\mu} \left| \frac{1}{(x-a)^{\mu-1}} \right|_{a+\epsilon}^b \right\} \\ &= \frac{1}{1-\mu} \cdot \frac{1}{(b-a)^{\mu-1}} - \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{\mu-1}} \end{aligned}$$

which tends to $\frac{1}{1-\mu} \cdot \frac{1}{(b-a)^{\mu-1}}$ or ∞ according as $\mu < 1$ or $\mu > 1$.

$$\begin{aligned} \text{If } \mu = 1, \quad \int_a^b \frac{dx}{x-a} &= \lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b \frac{dx}{x-a} = \lim_{\epsilon \rightarrow 0} \left| \log(x-a) \right|_{a+\epsilon}^b \\ &= \log(b-a) - \lim_{\epsilon \rightarrow 0} \log \epsilon \end{aligned}$$

which tends to infinity as $\epsilon \rightarrow 0$.

Thus $\int_a^b \frac{dx}{(x-a)^\mu}$ converges only when $\mu < 1$.

(c) $\int_a^b \frac{dx}{(b-x)^\mu}$ exists if $\mu < 1$ and does not exist if $\mu \geq 1$.

$\int_a^b \frac{dx}{(b-x)^\mu}$ has an infinite discontinuity at the upper limit b .

$$\begin{aligned}
 \text{If } \mu \neq 1, \quad \int_a^b \frac{dx}{(b-x)^\mu} &= \lim_{\epsilon \rightarrow 0} \int_a^{b-\epsilon} \frac{dx}{(b-x)^\mu} \\
 &= \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{\mu-1} \left| \frac{1}{(b-x)^{\mu-1}} \right|_a^{b-\epsilon} \right\} \\
 &= \frac{1}{\mu-1} \left(\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{\mu-1}} \right) - \frac{1}{\mu-1} \cdot \frac{1}{(b-a)^{\mu-1}},
 \end{aligned}$$

which tends to $\frac{1}{1-\mu} \cdot \frac{1}{(b-a)^{\mu-1}}$ or to ∞ according as $\mu < 1$ or $\mu > 1$.

If $\mu = 1$,

$$\int_a^b \frac{dx}{b-x} = \lim_{\epsilon \rightarrow 0} \int_a^{b-\epsilon} \frac{dx}{b-x} = \log(b-a) - \lim_{\epsilon \rightarrow 0} \log \epsilon$$

which tends to infinity as $\epsilon \rightarrow 0$.

Hence $\int_a^b \frac{dx}{(b-x)^\mu}$ converges only if $\mu < 1$.

12.5. Analogy with infinite series.

If S_n the sum of the first n terms of an infinite series can be found, it is possible to determine the nature of the series as regards convergence by passing to the limit. When however it is difficult to find S_n or S_n cannot be found, there are tests by which we can determine whether the series is convergent or not. Similarly if for

the improper integral $\int_a^b f(x) dx$, the primitive of $f(x)$ can be found,

we can by taking limits determine the nature of the integral. If, however, the primitive of $f(x)$ cannot be found, there are tests by which we can determine whether the improper integral exists or not. We shall presently establish some of these tests.

It should be noted however that the value of the improper integral if it exists cannot be found by these tests. To determine the value, we must be able to find the primitive.

12'6. Transformations.

An improper integral of class II, that is one which has an infinite discontinuity at a point c between a and b , can be expressed as the sum of two integrals with infinities at the limits, for

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Now an improper integral with an infinity at a limit can be transformed into an improper integral of class I by the substitutions

(i) $x = a + \frac{1}{y}$ for infinity at the lower limit a ,

(ii) $x = b - \frac{1}{y}$ for infinity at the upper limit b .

Thus if $f(x) \rightarrow \infty$ when $x \rightarrow a$, $\int_a^b f(x) dx$ is transformed by the substitution (i) into

$$\int_{(b-a)^{-1}}^{\infty} \frac{1}{y^2} f\left(a + \frac{1}{y}\right) dy.$$

Again if $f(x) \rightarrow \infty$ when $x \rightarrow b$, $\int_a^b f(x) dx$ is transformed by the substitution (ii) into

$$\int_{(b-a)^{-1}}^{\infty} \frac{1}{y^2} f\left(b - \frac{1}{y}\right) dy.$$

Hence the tests for convergence of improper integrals of class II are the same as those for class I.

12'7. Absolute convergence.

An improper integral $\int_a^b f(x) dx$ is said to be *absolutely convergent*

if $\int_a^b |f(x)| dx$ is convergent.

Theorem. An absolutely convergent improper integral is convergent; in other words, if $\int_a^b |f(x)| dx$ exists, so does

$$\int_a^b f(x) dx.$$

Let
$$\begin{aligned} f_1(x) &= f(x) \text{ when } f(x) \geq 0, \\ &= 0 \text{ when } f(x) < 0; \\ \text{and } f_2(x) &= 0 \text{ when } f(x) > 0, \\ &= -f(x) \text{ when } f(x) \leq 0. \end{aligned}$$

Then $f(x) = f_1(x) - f_2(x)$ and $|f(x)| = f_1(x) + f_2(x)$.

It is plain that $f_1(x) \geq 0$, $f_2(x) \geq 0$ and that $f_1(x)$ and $f_2(x)$ are continuous when $f(x)$ is so.

$$\begin{aligned} \text{Now } \int_a^b |f(x)| dx &= \int_a^b \{f_1(x) + f_2(x)\} dx \\ &= \int_a^b f_1(x) dx + \int_a^b f_2(x) dx, \end{aligned}$$

If then $\int_a^b |f(x)| dx$ exists, both $\int_a^b f_1(x) dx$ and $\int_a^b f_2(x) dx$

must also exist, as the non-existence of either would imply the non-existence of $\int_a^b |f(x)| dx$, and this would contradict the hypothesis.

Therefore

$$\int_a^b f(x) dx = \int_a^b f_1(x) dx - \int_a^b f_2(x) dx$$

also exists. This proves the theorem.

12'8 A necessary and sufficient condition for convergence.

(a) Improper integrals of class I.

By definition

$$\int_a^\infty f(x) dx = \lim_{\xi \rightarrow \infty} \int_a^\xi f(x) dx = I \text{ say, if the improper integral}$$

is supposed to exist.

Then for any arbitrary positive ϵ , we can find ξ such that for $x \geq \xi$

$$\left| I - \int_a^x f(x) dx \right| < \frac{\epsilon}{2}.$$

If now $x_2 > x_1 > \xi$,

$$\begin{aligned} \left| \int_{x_1}^{x_2} f(x) dx \right| &= \left| \int_a^{x_2} f(x) dx - \int_a^{x_1} f(x) dx \right| \\ &= \left| \int_a^{x_2} f(x) dx - I + I - \int_a^{x_1} f(x) dx \right| \\ &\leq \left| I - \int_a^{x_2} f(x) dx \right| + \left| I - \int_a^{x_1} f(x) dx \right| \\ &< \epsilon, \end{aligned}$$

$$\text{or } \left| \int_{x_1}^{x_2} f(x) dx \right| \rightarrow 0 \text{ as } x_1, x_2 \rightarrow \infty.$$

This integral $\int_{x_1}^{x_2} f(x) dx$ is called the *singular* integral correspond-

ing to $\int_a^\infty f(x) dx$ and the necessary and sufficient condition for

convergence of $\int_a^\infty f(x) dx$ is that the singular integral $\left| \int_{x_1}^{x_2} f(x) dx \right|$

must tend to 0 as x_1 and x_2 tend to infinity.

(b) Improper integrals of class II.

Let $f(x)$ have an infinite discontinuity at $x=b$ and let it be bounded and integrable in $(a, b-\delta)$, where $\delta (\neq 0)$ is a positive number however small. If $\int_a^b f(x) dx$ be supposed to exist, by definition,

$$\int_a^b f(x) dx = \lim_{\delta \rightarrow 0} \int_a^{b-\delta} f(x) dx = I \text{ say.}$$

Corresponding to any arbitrary positive ϵ , there exists a positive number η such that for $\delta < \eta$,

$$\left| I - \int_a^{b-\delta} f(x) dx \right| < \frac{\epsilon}{2}.$$

If $\delta_2 < \delta_1 < \eta$,

$$\left| I - \int_a^{b-\delta_2} f(x) dx \right| < \frac{\epsilon}{2}, \quad \left| I - \int_a^{b-\delta_1} f(x) dx \right| < \frac{\epsilon}{2}.$$

Hence easily

$$\left| \int_a^{b-\delta_2} f(x) dx - \int_a^{b-\delta_1} f(x) dx \right| < \epsilon,$$

$$\text{or, } \left| \int_{b-\delta_1}^{b-\delta_2} f(x) dx \right| \rightarrow 0 \quad \text{as } \delta_1, \delta_2 \rightarrow 0.$$

The necessary and sufficient condition for convergence of $\int_a^b f(x) dx$ is that the singular integral $\left| \int_{b-\delta_1}^{b-\delta_2} f(x) dx \right|$ must tend to 0 as δ_1 and δ_2 tend to 0.

12.9. Tests for convergence (Class I).

(a) If $f(x) > 0$ for $x > a$ and $x_2 > x_1 > a$, we have

$$\int_a^{x_2} f(x) dx = \int_a^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx > \int_a^{x_1} f(x) dx.$$

$\therefore \phi(x) = \int_a^x f(x) dx$ is a monotone increasing function of x . If

now $\phi(x)$ is bounded above, i.e., $\phi(x) \leq k$ for all $x > a$

$\int_a^x f(x) dx$ must tend to a limit as $x \rightarrow \infty$.

$$\therefore \int_a^{\infty} f(x) dx = \lim_{X \rightarrow \infty} \int_a^X f(x) dx \text{ must be convergent.}$$

And as $f(x) > 0$ for $x > a$, the convergence is absolute.

If $\phi(x)$ is not bounded above, $\phi(x)$ must tend to infinity as $x \rightarrow \infty$.

$\therefore \int_a^{\infty} f(x) dx$ does not exist in this case.

(b) If $|f(x)| \leq |\phi(x)|$ when $x \geq x_0$, then

$\int_a^{\infty} f(x) dx$ converges absolutely whenever $\int_a^{\infty} \phi(x) dx$ is absolutely

convergent; and if $\int_a^{\infty} f(x) dx$ is not absolutely convergent,

$\int_a^{\infty} \phi(x) dx$ is also not absolutely convergent.

Since $\int_a^{\infty} f(x) dx = \int_a^{x_0} f(x) dx + \int_{x_0}^{\infty} f(x) dx$ and $\int_a^{x_0} f(x) dx$ is a

proper integral, we need consider only $\int_{x_0}^{\infty} f(x) dx$.

Now since the integral of a positive function is positive and increases as the range increases, both $\int_{x_0}^X |f(x)| dx$ and $\int_{x_0}^X |\phi(x)| dx$ are monotone increasing functions of X .

If $\int_a^{\infty} \phi(x) dx$ converges absolutely, then obviously

$$\lim_{X \rightarrow \infty} \int_{x_0}^X |\phi(x)| dx \text{ tends to a finite limit } l.$$

And since $|f(x)| \leq |\phi(x)|$ for $x \geq x_0$,

$$\int_{x_0}^X |f(x)| dx \leq \int_{x_0}^X |\phi(x)| dx \leq l.$$

$$\therefore \lim_{X \rightarrow \infty} \int_{x_0}^X |f(x)| dx \text{ tends to a finite limit } \leq l,$$

i.e., $\int_a^{\infty} f(x) dx$ is absolutely convergent.

If however $\int_a^X f(x) dx$ is not absolutely convergent, then

$\int_a^X |f(x)| dx$ does not tend to a finite limit as $X \rightarrow \infty$; and as

$\int_a^X |f(x)| dx$ is a monotonic increasing function of X , it tends to ∞ as $X \rightarrow \infty$.

$$\therefore \int_{x_0}^X |f(x)| dx \rightarrow \infty \text{ as } X \rightarrow \infty.$$

$$\text{Hence } \int_{x_0}^X |\phi(x)| dx \geq \int_{x_0}^X |f(x)| dx \rightarrow \infty \text{ as } X \rightarrow \infty,$$

$$\text{i.e., } \int_a^\infty \phi(x) dx \text{ is not absolutely convergent.}$$

(c) If $\phi(x)$ is positive for $x > a$ and if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\phi(x)} = l$$

where l is finite and $\neq 0$, then the integrals $\int_a^\infty f(x) dx$ and $\int_a^\infty \phi(x) dx$

are both absolutely convergent or are both divergent.

Suppose $l > 0$; choose ϵ so that $0 < \epsilon < l$. Then

$$\left| \frac{f(x)}{\phi(x)} - l \right| < \epsilon \text{ when } x \geq x',$$

$$\text{i.e., } (l - \epsilon)\phi(x) < f(x) < (l + \epsilon)\phi(x). \quad \dots \quad (1)$$

Now by hypothesis $\phi(x) > 0$ when $x \geq x'$; therefore $f(x)$ is also positive when $x \geq x'$.

$$\therefore \int_{x'}^X f(x) dx \text{ and } \int_{x'}^X \phi(x) dx \text{ are both monotonic functions of}$$

X ; they must either converge or diverge but cannot oscillate.

From (1) we get by integration.

$$(l - \epsilon) \int_{x'}^X \phi(x) dx < \int_{x'}^X f(x) dx < (l + \epsilon) \int_{x'}^X \phi(x) dx \quad \dots \quad (2)$$

From the left-half of the inequalities (2), we obtain :

$$\text{If } \int_a^\infty f(x) dx \text{ is convergent, so is } \int_a^\infty \phi(x) dx \text{ and if } \int_a^\infty \phi(x) dx \text{ is}$$

divergent, so also is $\int_a^\infty f(x) dx$.

Again the right-half of the inequalities (2) gives us the results :

If $\int_a^\infty \phi(x) dx$ is convergent, so is $\int_a^\infty f(x) dx$ and if $\int_a^\infty f(x) dx$ is

divergent, $\int_a^\infty \phi(x) dx$ is also divergent.

Thus subject to the conditions stated both $\int_a^\infty f(x) dx$ and $\int_a^\infty \phi(x) dx$

diverge together or converge together ; and since $\phi(x)$ and $f(x)$ have the same sign when $x \geq x'$, the convergence is absolute.

When $l < 0$, we take $l = -l'$ so that $l' > 0$. Now

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\phi(x)} = l = -l', \quad \text{or} \quad \lim_{x \rightarrow \infty} \frac{-f(x)}{\phi(x)} = l'.$$

Thus $-\int_a^\infty f(x) dx$ and $\int_a^\infty \phi(x) dx$ behave alike and therefore

$\int_a^\infty f(x) dx$ and $\int_a^\infty \phi(x) dx$ behave alike.

When $l = 0$, we can find x' such that $\frac{|f(x)|}{\phi(x)} < \epsilon$, i.e., $|f(x)|$

$< \epsilon \cdot \phi(x)$ when $x \geq x'$. Hence if $\int_a^\infty \phi(x) dx$ converges, $\int_a^\infty f(x) dx$

converges absolutely.

When $l = \infty$, we can find M such that $\frac{f(x)}{\phi(x)} > M$ or $f(x)$

$> M\phi(x)$ when $x \geq x'$. Hence $\int_a^\infty f(x) dx$ diverges if $\int_a^\infty \phi(x) dx$

diverges.

Similarly when $\frac{f(x)}{\phi(x)} \rightarrow -\infty$.

(d) μ -test.

We know that if $a > 0$ and $\mu > 1$, $\int_a^\infty \frac{dx}{x^\mu}$ converges absolutely

[Art. 12'4 (a)]. Let $f(x)$ be integrable when $x \geq a$. In (c) take $\left[\phi(x) = \frac{1}{x^\mu}; \text{ then } \frac{f(x)}{\phi(x)} = x^\mu f(x) \right]$. If now

$$\lim_{x \rightarrow \infty} [x^\mu f(x)] = l \quad (l \neq 0 \text{ or } \pm \infty) \text{ and } \mu > 1,$$

then $\int_a^\infty f(x) dx$ converges absolutely.

12'10. Test for convergence (Class II).

We refer to the remark in Art. 12'6 that the tests for convergence of improper integrals of Class II are the same as those for Class I.

We state here only two of these tests.

(a) Let $f(x)$ and $\phi(x)$ be integrable functions when $a < x \leq b$ and $\phi(x) > 0$ when $x \geq a$; then if

$$\lim_{x \rightarrow a+0} \frac{f(x)}{\phi(x)} = l \quad (l \neq 0),$$

the integrals $\int_a^b f(x) dx$ and $\int_a^b \phi(x) dx$ both converge absolutely or both diverge.

If $l=0$ and $\int_a^b \phi(x) dx$ converges, then $\int_a^b f(x) dx$ converges absolutely.

If $l=\pm\infty$ and $\int_a^b \phi(x) dx$ diverges, then $\int_a^b f(x) dx$ also diverges.

(b) μ -test for infinity at the lower limit.

If $f(x)$ be an integrable function in $(a+\epsilon, b)$, where $0 < \epsilon < b-a$,

then $\int_a^b f(x) dx$ converges absolutely if $\lim_{x \rightarrow a+0} [(x-a)^\mu f(x)] = l$

($l \neq 0$ or $\pm \infty$) and $0 < \mu < 1$; it diverges when $\mu \geq 1$.

We have a similar μ -test for infinity at the upper limit.

12.11. We give below without proof two tests which are sometimes useful.

(a) **Abel's test**: Let $\phi(x)$ be bounded and monotonic in (a, ∞) and let $\int_a^\infty f(x) dx$ be convergent; then $\int_a^\infty f(x) \phi(x) dx$ is convergent.

(b) **Dirichlet's test**: Let $\phi(x)$ be bounded and monotonic in (a, ∞) and let $\phi(x) \rightarrow 0$ when $x \rightarrow \infty$; further let $\int_a^\xi f(x) dx$ be bounded when $\xi \geq a$. Then $\int_a^\infty f(x) \phi(x) dx$ is convergent.

Examples:

1. Show that the following improper integrals are convergent. Find the value of each.

$$(a) \int_1^\infty \frac{dx}{(1+x)\sqrt{x}} \quad (C. H. 1956, 1959, 1963)$$

$$(b) \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \quad (C. H. 1960)$$

$$(c) \int_{a-1}^{a+1} \frac{dx}{(x-a)^{\frac{1}{3}}} \quad (C. H. 1959)$$

[Hint. Find indefinite integrals and take limits.

$$(a) \frac{\pi}{2}, (b) \pi, (c) 0.]$$

2. Show that $\int_0^1 \frac{dx}{x(1+x)}$ is divergent.

(C. U. 1960)

[Hint. Follow the same method.]

3. Examine the convergence of

$$(a) \int_a^\infty \frac{dx}{x\sqrt{1+x^2}}.$$

$$(b) \int_0^\infty e^{-x^2} dx.$$

$$(c) \int_a^\infty \frac{\sin^2 x}{x^2} dx.$$

[Hint. In (a) and (b), use the μ -test. For (c) $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ when $x \geq a > 0$.]

4. Use the μ -test to show that the following integrals are convergent.

$$(a) \int_1^\infty \frac{x dx}{(1+x)^3}.$$

$$(b) \int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x)^2}.$$

$$(c) \int_{\frac{1}{2}}^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}}.$$

$$(d) \int_0^{\frac{\pi}{2}} \frac{\sqrt{x}}{\sin x} dx.$$

5. Apply the same test to prove that the following integrals are non-convergent.

$$(a) \int_1^\infty \frac{dx}{x^{\frac{1}{3}}(1+x)^{\frac{1}{2}}}.$$

$$(b) \int_0^1 \frac{dx}{x^2(1+x)^2}.$$

$$(c) \int_0^1 \frac{dx}{[(x-1)^6(x-2)]^{\frac{1}{5}}}.$$

6. Discuss the convergence of

$$\int_0^1 \frac{x^{a-1}}{1+x} dx.$$

[If $a \geq 1$, the integral is proper. If $a < 1$, there is an infinity at $x=0$; use the μ -test taking $x^\mu = x^{1-a}$. It converges when $1 - a < 1$, i.e., $a > 0$.]

7. Show that $\int_0^{\frac{\pi}{2}} x^p \operatorname{cosec}^q x \, dx$ is convergent if $q < p+1$.

(C. H. 1961)

$$[f(x) = \frac{x^p}{(\sin x)^q} = \left(\frac{x}{\sin x}\right)^q \cdot x^{p-q} = \left(\frac{x}{\sin x}\right)^q \cdot \frac{1}{x^{q-p}}. \text{ At } x=0,$$

$f(x) \rightarrow 0$ if $p-q > 0$ and $f(x) \rightarrow \infty$ if $p-q < 0$. Thus the integral is proper if $p-q > 0$ and improper if $p-q < 0$; and $x=0$ is the only point of discontinuity in the latter case. Take $\mu = q-p$; the integral converges if $q-p < 1$ or $q < p+1$.]

8. Prove that $\int_0^{\infty} e^{-ax} \cos bx \, dx$ is convergent. (C. H. 1960)

Show also that $\int_0^{\infty} e^{-ax} \sin bx \, dx$ is convergent.

[Their values are respectively $\frac{a}{a^2+b^2}$, $\frac{b}{a^2+b^2}$, ($a > 0$)].

9. Prove that $\int_0^{\infty} e^{-x^2} \, dx$ is convergent and that its value

is less than $\frac{\pi}{2}$. (C. H. 1960, '68)

[By the μ -test, taking $\mu = 2$, $\operatorname{Lt} [x^2 e^{-x^2}] = 0$ and as $\mu = 2 > 1$, the integral converges. Now $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \dots > 1 + x^2$;

$$e^{-x^2} < \frac{1}{1+x^2}.$$

$$\text{So } \int_0^{\infty} e^{-x^2} \, dx < \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.]$$

10. Show that both $\int_0^{\infty} \frac{\log x \, dx}{x^2}$ and $\int_0^{\pi} \frac{dx}{1 - \cos x}$ diverge.

[Hint. Find the indefinite integral in each case.] (C. H. 1956)

11. If $a > 1$, show that the integral

$$\int_a^{\infty} \frac{dx}{x(\log x)^{1+\mu}}$$

converges if $\mu > 0$ but diverges if $\mu \leq 0$. Find the value of the integral in the case of convergence. (C. H. 1964)

12. Show that $\int_0^{\infty} \frac{x^2 \, dx}{(a^2 + x^2)^2}$ is convergent. (C. H. 1964)

[Hint. Put $x = a \tan \theta$.]

13. Prove that $\int_0^{\infty} \cos x^2 \, dx$ is convergent. (C. H. 1962)

$$\left[\int_0^{\infty} \cos x^2 \, dx = \int_0^1 \cos x^2 \, dx + \int_1^{\infty} \cos x^2 \, dx ; \text{ now } \int_0^1 \cos x^2 \, dx \right.$$

is proper. For the other part, integrate by parts the integral

$$\int_1^X \cos x^2 \, dx \text{ writing it as } \int_1^X \frac{\cos x^2}{2x} \cdot 2x \, dx, \text{ and take limits.}$$

$$\left[\int_1^{\infty} \cos x^2 \, dx \text{ will be found to be convergent.} \right]$$

14. Show that $\int_0^{\infty} \sin x^2 \, dx$ is convergent.

15. Show that $\int_0^{\infty} \frac{x}{e^x + 1} \, dx$ is convergent and find its value.

(C. H. 1962)

[*Hint.* $\frac{x}{e^x+1} = \frac{x e^{-x}}{1+e^{-x}} = x e^{-x} (1+e^{-x})^{-1}$; expand, integrate, and put limits; then use the result. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$.

The value is $\frac{\pi^2}{12}$.]

16. Prove that $\int_0^{\frac{\pi}{2}} \sin x \log \sin x \, dx$ exists and find its value. (C. H. 1962)

[The only infinity is at $x=0$. Integrating by parts

$$\begin{aligned} \int_{\epsilon}^{\frac{\pi}{2}} \sin x \log \sin x \, dx &= \left[-\cos x \log \sin x + \log \tan \frac{x}{2} + \cos x \right]_{\epsilon}^{\frac{\pi}{2}} \\ &= \cos \epsilon \log \sin \epsilon - \log \tan \frac{\epsilon}{2} - \cos \epsilon, \end{aligned}$$

as the right side vanishes for $x = \frac{\pi}{2}$.

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{2}} \sin x \log \sin x \, dx &= \lim_{\epsilon \rightarrow 0} \left\{ \cos \epsilon \log \sin \epsilon - \log \tan \frac{\epsilon}{2} - \cos \epsilon \right\} \\ &= \lim_{\epsilon \rightarrow 0} \left[\cos \epsilon \left\{ \log \sin \frac{\epsilon}{2} + \log \left(2 \cos \frac{\epsilon}{2} \right) \right\} - \log \sin \frac{\epsilon}{2} + \log \cos \frac{\epsilon}{2} - \cos \epsilon \right] \\ &= \lim_{\epsilon \rightarrow 0} \left[-\left(1 - \cos \epsilon \right) \log \sin \frac{\epsilon}{2} + \cos \epsilon \log \left(2 \cos \frac{\epsilon}{2} \right) + \log \cos \frac{\epsilon}{2} - \cos \epsilon \right] \end{aligned}$$

Now when $\epsilon \rightarrow 0$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left[\left(1 - \cos \epsilon \right) \log \sin \frac{\epsilon}{2} \right] &= \lim_{u \rightarrow 0} \left(2u^2 \log u \right) = 0 \\ &\quad \left(\text{where } u = \sin \frac{\epsilon}{2} \right) \end{aligned}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin x \log \sin x \, dx = \log 2 - 1 = \log \left(\frac{2}{e} \right).$$

17. Evaluate $\int_0^a x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}} \, dx$. (C. H. 1966)

[Put $x^2 = a^2 \cos \theta$; on integrating and putting limits, the value is found to be $\frac{a^2}{2} \left(\frac{\pi}{2} - 1 \right)$.]

18. Show that $\int_0^{\frac{\pi}{2}} (\cos x)^l (\sin x)^m dx$ is convergent if $l > -1$, $m > -1$. (C. H. 1966)

[The integral is proper when $l \geq 0, m \geq 0$. Consider the case $l < 0, m < 0$. Expressing $\int_0^{\frac{\pi}{2}} (\cos x)^l (\sin x)^m dx$

$$= \int_0^{\frac{\pi}{4}} (\cos x)^l (\sin x)^m dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x)^l (\sin x)^m dx$$

$$= u_1 + u_2 \text{ say. } u_1 \text{ has an infinity at } x=0 \text{ and } u_2 \text{ at } x=\frac{\pi}{2}.$$

For u_1 , take $f(x) = (\cos x)^l (\sin x)^m$ and $\phi(x) = x^m$.

$$\text{Then } \lim_{x \rightarrow 0} \frac{f(x)}{\phi(x)} = \lim_{x \rightarrow 0} (\cos x)^l \left(\frac{\sin x}{x} \right)^m = 1.$$

$$\text{Now } \int_0^{\frac{\pi}{4}} \phi(x) dx = \int_0^{\frac{\pi}{4}} x^m dx = \int_0^{\frac{\pi}{4}} \frac{1}{x^{-m}} dx \text{ is convergent when}$$

$-m < 1$, i.e., $m > -1$. Similarly for u_2 .]

19. Discuss the convergence of $\int_0^{\infty} \frac{dx}{1+x^2 \sin^2 x}$ (C. H. 1961)

[In the range $n\pi < x < (n+1)\pi$, we have since the integrand is positive

$$\frac{1}{1+(n+1)^2 \pi^2 \sin^2 x} < \frac{1}{1+x^2 \sin^2 x} < \frac{1}{1+n^2 \pi^2 \sin^2 x}.$$

$$\begin{aligned} \text{Now } \int \frac{dx}{1+A \sin^2 x} &= \int \frac{dx}{\cos^2 x + (1+A) \sin^2 x} = \int \frac{\sec^2 x dx}{1+(1+A) \tan^2 x} \\ &= \frac{1}{(1+A)^{\frac{1}{2}}} \tan^{-1} \left\{ (1+A)^{\frac{1}{2}} \tan x \right\}. \end{aligned}$$

Further

$$\begin{aligned}
 I &= \int_{n\pi}^{(n+1)\pi} \frac{dx}{1+A \sin^2 x} = \int_0^\pi \frac{dy}{1+A \sin^2 y} \quad (\text{on putting } x=n\pi+y) \\
 &= \int_0^{\frac{\pi}{2}} \frac{dy}{1+A \sin^2 y} + \int_{\frac{\pi}{2}}^\pi \frac{dy}{1+A \sin^2 y} = u_1 + u_2 \text{ say.}
 \end{aligned}$$

On putting $y = z + \frac{\pi}{2}$, u_2 transforms into

$$\int_0^{\frac{\pi}{2}} \frac{dz}{1+A \cos^2 z} = \int_0^{\frac{\pi}{2}} \frac{dx}{1+A \sin^2 x}.$$

$$\begin{aligned}
 \text{Thus } I &= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1+A \sin^2 x} = \frac{2}{\sqrt{1+A}} \left| \tan^{-1} \left(\sqrt{1+A} \tan x \right) \right|_0^{\frac{\pi}{2}} \\
 &= \frac{\pi}{\sqrt{1+A}}.
 \end{aligned}$$

$$\text{Hence if } J = \int_0^\infty \frac{dx}{1+x^2 \sin^2 x},$$

$$\sum \frac{\pi}{\{1+(n+1)^2 \pi^2\}^{\frac{1}{2}}} < J < \sum \frac{\pi}{\{1+n^2 \pi^2\}^{\frac{1}{2}}}.$$

Thus J converges or diverges with $\sum \frac{\pi}{(1+n^2 \pi^2)^{\frac{1}{2}}}$ and by com-

paring it with $\sum \frac{1}{n\pi}$, it is easily seen that J diverges. Hence the given integral is divergent.

Note. $\int_0^\infty \frac{dx}{1+x^4 \sin^2 x^2}$ converges.]

20. Prove that $\int_0^\infty \frac{dx}{(x + \sqrt{1+x^2})^n} = \frac{n}{n^2-1} \quad (n > 1). \quad (C. H. 1964)$

[As $(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x)=1$, we at once have

$$\int_0^{\infty} \frac{dx}{(\sqrt{x^2+1}+x)^n} = \int_0^{\infty} (\sqrt{x^2+1}-x)^n dx.$$

We may put $x=\sinh \theta$, or proceed as follows.

Put $\sqrt{x^2+1}-x = \frac{1}{\sqrt{x^2+1}+x} = y$; the limits $0, \infty$ are changed into $1, 0$.

Now from $\sqrt{x^2+1}-x=y$ and $\sqrt{x^2+1}+x=\frac{1}{y}$,

$$\text{we get } \sqrt{x^2+1} = \frac{1}{2} \left(y + \frac{1}{y} \right).$$

$$\text{Also } dy = \left(\frac{x}{\sqrt{x^2+1}} - 1 \right) dx = -\frac{y}{\sqrt{x^2+1}} dx,$$

$$\text{or, } dx = -\frac{1}{2} \left(y + \frac{1}{y} \right) \frac{dy}{y} = -\frac{1}{2} \left(1 + \frac{1}{y^2} \right) dy.$$

$$\begin{aligned} \therefore \int_0^{\infty} (\sqrt{x^2+1}-x)^n dx &\text{ is transformed into } \frac{1}{2} \int_1^0 \left(1 + \frac{1}{y^2} \right) y^n dy \\ &= \frac{1}{2} \int_0^1 (y^n - y^{n-2}) dy = \frac{1}{2} \left[\frac{y^{n+1}}{n+1} + \frac{y^{n-1}}{n-1} \right]_0^1 = \frac{1}{2} \left(\frac{1}{n+1} + \frac{1}{n-1} \right) = \frac{n}{n^2-1}. \end{aligned}$$

$$\text{Thus } \int_0^{\infty} \frac{dx}{(\sqrt{x^2+1}+x)^n} = \frac{n}{n^2-1}.$$

$$21. \text{ Show that } \int_0^{\frac{\pi}{2}} \log \sin x \, dx \text{ converges to } -\frac{\pi}{2} \log 2.$$

[The only infinity is at $x=0$. We know that $\int_0^{\frac{\pi}{2}} \frac{dx}{x^\mu}$ converges

if $0 < \mu < 1$. Take $f(x)=\log \sin x$ and $\phi(x)=\frac{1}{x^\mu}$ ($\mu < 1$), we get

$$\frac{f(x)}{\phi(x)} = x^\mu \log \sin x. \text{ Now}$$

$$\begin{aligned} \text{Lt}_{x \rightarrow 0} \left(\frac{\log \sin x}{x^{-\mu}} \right) \text{ of form } \frac{\infty}{\infty} &= \text{Lt}_{x \rightarrow 0} \left(\frac{\frac{\cos x}{\sin x}}{-\mu x^{-\mu-1}} \right) \\ &= \text{Lt}_{x \rightarrow 0} \left(-\frac{1}{\mu} x^{\mu} \frac{x}{\sin x} \cos x \right) = 0. \end{aligned}$$

And as $\int_0^{\frac{\pi}{2}} \frac{dx}{x^{\mu}}$ converges, it follows that $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ is convergent.

$$\text{We have } \int_0^{\pi} \log \sin x \, dx = \int_0^{\frac{\pi}{2}} \log \sin x \, dx + \int_{\frac{\pi}{2}}^{\pi} \log \sin x \, dx.$$

In $\int_{\frac{\pi}{2}}^{\pi} \log \sin x \, dx$, put $x = \pi - z$; it transforms into

$$\int_0^{\frac{\pi}{2}} \log \sin x \, dx.$$

$$\therefore \int_0^{\pi} \log \sin x \, dx = 2 \int_0^{\frac{\pi}{2}} \log \sin x \, dx. \quad \dots \quad \dots (1)$$

Now $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$,

$$\begin{aligned} \therefore \int_0^{\pi} \log \sin x \, dx &= \pi \log 2 + \int_0^{\pi} \log \sin \frac{x}{2} \, dx + \int_0^{\pi} \log \cos \frac{x}{2} \, dx \\ &= \pi \log 2 + 2 \int_0^{\frac{\pi}{2}} \log \sin x \, dx + 2 \int_0^{\frac{\pi}{2}} \log \cos x \, dx \end{aligned}$$

or by (1)

$$2 \int_0^{\frac{\pi}{2}} \log \sin x \, dx = \pi \log 2 + 4 \int_0^{\frac{\pi}{2}} \log \sin x \, dx$$

$$\text{whence } \int_0^{\frac{\pi}{2}} \log \sin x \, dx = -\frac{\pi}{2} \log 2.$$

$$\text{Cor. } \int_0^{\pi} x \log \sin x \, dx = -\frac{\pi^2}{2} \log 2.]$$

22. Evaluate

$$(a) \int_0^{\pi} \log (1 - \cos x) \, dx. \quad (C. H. 1966)$$

$$(b) \int_0^{\infty} \log \left(x + \frac{1}{x} \right) \frac{dx}{1+x^2}. \quad (C. H. 1964, 1966)$$

[(a) $\frac{\pi}{2} \log \frac{1}{2}$. In (b), put $x = \tan \theta$, the value is $\pi \log 2$.]

$$23. \text{ Prove that } \int_0^{\infty} \frac{\sin x}{x} \, dx \text{ is convergent.}$$

(C. H. 1960, 1963, 1965, 1968)

[Since the integrand $\rightarrow 1$ as $x \rightarrow 0$, 0 is not a point of infinite discontinuity.

We have seen that the necessary and sufficient condition for the convergence of $\int_a^{\infty} f(x) \, dx$ is that the corresponding singular integral

$\int_{x'}^{x''} f(x) \, dx$ must tend to 0 as x', x'' tend to infinity. [Art. 12'8 (a)]

The singular integral corresponding to $\int_0^{\infty} \frac{\sin x}{x} dx$ is $\int_{x'}^{x''} \frac{\sin x}{x} dx$
 $(0 < x' < x'').$

Taking $X < x' < x''$ where X is arbitrarily large and applying the Second Mean Value theorem to $\int_{x'}^{x''} \frac{\sin x}{x} dx$, we have

$$\int_{x'}^{x''} \frac{\sin x}{x} dx = \frac{1}{x'} \int_{x'}^{\xi} \sin x dx + \frac{1}{x''} \int_{\xi}^{x''} \sin x dx$$

where $0 < X \leq x' \leq \xi \leq x''$.

But $\left| \int_{x'}^{\xi} \sin x dx \right|$ and $\left| \int_{\xi}^{x''} \sin x dx \right|$ are each less than or equal to 2.

$$\therefore \left| \int_{x'}^{x''} \frac{\sin x}{x} dx \right| \leq 2 \left(\frac{1}{x'} + \frac{1}{x''} \right) < \frac{4}{x'}.$$

Thus $\left| \int_{x'}^{x''} \frac{\sin x}{x} dx \right| < \epsilon$ when $x'' > x' \geq X$, provided that

$$X > \frac{4}{\epsilon}.$$

Therefore $\int_0^{\infty} \frac{\sin x}{x} dx$ converges.]

24. Prove that $\int_0^{\infty} \frac{\sin x}{x} dx$ is not absolutely convergent.

[We have to prove that $\int_0^{\infty} \frac{|\sin x|}{x} dx$ diverges. We first con-

sider the integral $\int_0^{n\pi} \frac{|\sin x|}{x} dx$, where n is any positive integer.

$$\text{We have } \int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{r=1}^n \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx.$$

$$\text{But } \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi + y} dy, \text{ on putting } x = (r-1)\pi + y;$$

and since $r\pi$ is the maximum value of $(r-1)\pi + y$ in $(0, \pi)$, we get

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx \geq \frac{1}{r\pi} \int_0^{\pi} \sin y dy = \frac{2}{r\pi}.$$

$$\therefore \int_0^{n\pi} \frac{|\sin x|}{x} dx \geq \sum_{r=1}^n \frac{2}{r\pi} = \frac{2}{\pi} \sum_{r=1}^n \frac{1}{r}.$$

But since $\sum_{r=1}^n \frac{1}{r} \rightarrow \infty$ as $n \rightarrow \infty$, we find that

$$\int_0^{n\pi} \frac{|\sin x|}{x} dx \rightarrow \infty, \text{ as } n \rightarrow \infty.$$

If now $X \geq n\pi$ where X is a real number,

$$\int_0^X \frac{|\sin x|}{x} dx \geq \int_0^{n\pi} \frac{|\sin x|}{x} dx.$$

As $X \rightarrow \infty$, n also $\rightarrow \infty$. Thus

$$\int_0^X \frac{|\sin x|}{x} dx \rightarrow \infty \text{ as } X \rightarrow \infty$$

so that $\int_0^\infty \frac{|\sin x|}{x} dx$ does not converge.]

$$25. \text{ Show that } \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}. \quad (C. H. 1968)$$

$$\left[\text{Let } u_n = \int_0^{\frac{\pi}{2}} \sin 2nx \cot x dx. \right.$$

Since $\sin (2n+1)x = \sin 2nx \cos x + \cos 2nx \sin x$, we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin 2nx \cos x}{\sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{\sin (2n+1)x}{\sin x} dx - \int_0^{\frac{\pi}{2}} \cos 2nx dx \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin (2n+1)x}{\sin x} dx \text{ as } \int_0^{\frac{\pi}{2}} \cos 2nx dx = 0. \end{aligned}$$

$$\text{Now } \int_0^{\frac{\pi}{2}} \frac{\sin (2n+1)x - \sin (2n-1)x}{\sin x} dx = \int_0^{\frac{\pi}{2}} 2 \cos 2nx dx = 0.$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin (2n+1)x}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin (2n-1)x}{\sin x} dx = \dots = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$

$$\text{Thus } u_n = \int_0^{\frac{\pi}{2}} \sin 2nx \cot x dx = \frac{\pi}{2}.$$

$$\text{Next let } v_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x} dx. \text{ Putting } 2nx = z, \text{ we get}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{x} dx = \int_0^{n\pi} \frac{\sin z}{z} dz.$$

As $n \rightarrow \infty$, $v_n \rightarrow v = \int_0^{\infty} \frac{\sin x}{x} dx$.

Now $u_n - v_n = \int_0^{\frac{\pi}{2}} \sin 2nx \left(\cot x - \frac{1}{x} \right) dx$ and on putting $2nx = z$,

$$\begin{aligned} \text{we have } u_n - v_n &= \int_0^{n\pi} \sin z \left\{ \frac{\cos \frac{z}{2n}}{\sin \frac{z}{2n}} - \frac{2n}{z} \right\} \frac{dz}{2n} \\ &= \int_0^{n\pi} \frac{\sin z}{z} \left\{ \cos \frac{z}{2n} \cdot \frac{\frac{z}{2n}}{\sin \frac{z}{2n}} - 1 \right\} dz. \end{aligned}$$

$$\text{As } n \rightarrow \infty, \left\{ \cos \frac{z}{2n} \cdot \frac{\frac{z}{2n}}{\sin \frac{z}{2n}} - 1 \right\} \rightarrow 0, \therefore u_n - v_n \rightarrow 0.$$

Now $\lim_{n \rightarrow \infty} (u_n - v_n) = \frac{\pi}{2} - v = 0. \therefore v = \frac{\pi}{2}.$

Thus $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}. \quad]$

Cor. $\int_0^{\infty} \frac{\sin ax}{x} dx = \frac{\pi}{2}, 0, \text{ or, } -\frac{\pi}{2}$ according as $a > 0, = 0,$
or < 0 .

26. (a) Show that $\int_a^{\infty} \frac{\sin^2 x}{x^2} dx$ is convergent.

(b) Prove that $\int_0^{\infty} \frac{\sin^2 x}{x} dx = \frac{\pi}{2}.$

(c) Evaluate $\int_{-\infty}^{\infty} \frac{\sin^2 mx}{x^2} dx.$ (C. H 1953)

[(a) Observe that $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}.$

(b) Integrate by parts.

(c) $\int_{-\infty}^{\infty} \frac{\sin^2 mx}{x^2} dx = 2 \int_0^{\infty} \frac{\sin^2 mx}{x^2} dx.$ The value is $m\pi.$]

27. Prove that $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$ if $0 < p < 1.$

[Near $x=0$, the integrand may be written $\frac{1}{1+x} \cdot \frac{1}{x^{1-p}}$; hence by the μ -test for convergency at $x=0$, $1-p < 1$ or, $p > 0$. Near $x=\infty$, it may be written as $\frac{1}{1+\frac{1}{x}} \cdot \frac{1}{x^{2-p}}$; hence as above for convergency at $x=\infty$, $2-p > 1$, i.e., $p < 1$. Thus for convergency at both limits $0 < p < 1$.

$$I = \int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \int_0^1 \frac{x^{p-1}}{1+x} dx + \int_1^{\infty} \frac{x^{p-1}}{1+x} dx = u_1 + u_2 \text{ say.}$$

Putting $x = \frac{1}{y}$ in u_2 , we have $\int_1^{\infty} \frac{x^{p-1}}{1+x} dx = \int_0^1 \frac{y^{-p}}{1+y} dy = \int_0^1 \frac{x^{-p}}{1+x} dx.$

$$\therefore I = \int_0^1 \frac{x^{p-1} + x^{-p}}{1+x} dx.$$

Now $\frac{1}{1+x} = \sum_{k=0}^{k=n} (-1)^k x^k + (-1)^{n+1} \frac{x^{n+1}}{1+x};$

$$\text{but } \int_0^1 (x^{p-1} + x^{-p}) x^k dx = \frac{1}{k+p} + \frac{1}{k+1-p},$$

so that

$$I = \sum_{k=0}^{k=n} (-1)^k \left(\frac{1}{k+p} + \frac{1}{k+1-p} \right) + (-1)^{n+1} \int_0^1 \frac{(x^{p-1} + x^{-p}) x^{n+1}}{1+x} dx.$$

As $n \rightarrow \infty$, the integral last written converges to 0, because it is less than

$$\int_0^1 (x^{p-1} + x^{-p}) x^{n+1} dx = \frac{1}{n+p+1} + \frac{1}{n-p+2}.$$

$$\text{Hence } I = \sum_0^{\infty} (-1)^k \left(\frac{1}{k+p} + \frac{1}{k+1-p} \right).$$

Now from any text-book on Higher Trigonometry, it will be seen that

$$\begin{aligned} \frac{1}{\sin x} &= \frac{1}{x} + \sum_1^{\infty} (-1)^n \left(\frac{1}{x-n\pi} + \frac{1}{x+n\pi} \right) \\ &= \sum_0^{\infty} (-1)^n \left(\frac{1}{x+n\pi} + \frac{1}{(n+1)\pi-x} \right). \end{aligned}$$

Putting $x = p\pi$ where p is a proper fraction, we find

$$\frac{\pi}{\sin p\pi} = \sum_0^{\infty} (-1)^n \left(\frac{1}{n+p} + \frac{1}{n+1-p} \right).$$

Hence finally

$$I = \int_0^1 \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi} \cdot]$$

28. Show that if $n \geq 3$, the integral $I_n = \int_0^\infty \frac{dx}{\cosh^n x}$ is convergent and is equal to $\frac{n-2}{n-1} I_{n-2}$ and deduce that $I_5 = \frac{3\pi}{16}$.
(C. H. 1964)

$$\begin{aligned} \left[\int \frac{dx}{\cosh^n x} \right] &= \int \operatorname{sech}^{n-2} x \operatorname{sech}^2 x \, dx \\ &= \operatorname{sech}^{n-2} x \tanh x - \int (n-2) \operatorname{sech}^{n-3} x (-\operatorname{sech} x \tanh x) \tanh x \, dx \\ &= \operatorname{sech}^{n-2} x \tanh x + (n-2) \int \operatorname{sech}^{n-2} x \tanh^2 x \, dx \\ &= \operatorname{sech}^{n-2} x \tanh x + (n-2) \int \operatorname{sech}^{n-2} x (1 - \operatorname{sech}^2 x) \, dx \\ &= \operatorname{sech}^{n-2} x \tanh x + (n-2) \int \operatorname{sech}^{n-2} x \, dx - (n-2) \int \operatorname{sech}^n x \, dx. \\ \therefore \int \operatorname{sech}^n x \, dx &= \frac{\operatorname{sech}^{n-2} x \tanh x}{n-1} + \frac{n-2}{n-1} \int \operatorname{sech}^{n-2} x \, dx. \end{aligned}$$

$$\text{Hence } I_n = \left[\frac{\operatorname{sech}^{n-2} x \tanh x}{n-1} \right]_0^\infty + \frac{n-2}{n-1} I_{n-2} = \frac{n-2}{n-1} I_{n-2} \text{ as}$$

$\operatorname{sech}^{n-2} x \tanh x$ vanishes for both limits.

$$\therefore I_n = \frac{n-2}{n-1} I_{n-2}.$$

(i) If n be odd, I_n can be made to depend on I_1 , and

$$I_1 = \int_0^\infty \operatorname{sech} x \, dx = \left[2 \tan^{-1}(e^x) \right]_0^\infty = 2 \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{2}.$$

$\therefore I_n$ converges when n is odd.

(ii) If n be even, I_n can be made to depend on I_2 , and

$$I_2 = \int_0^\infty \operatorname{sech}^2 x \, dx = \left[\tanh x \right]_0^\infty = \left[\frac{e^x - e^{-x}}{e^x + e^{-x}} \right]_0^\infty = \left[\frac{1 - e^{-2x}}{1 + e^{-2x}} \right]_0^\infty = 1.$$

$\therefore I_n$ converges when n is even.

Thus I_n converges for all positive integral values of $n \geq 3$.

Also from $I_n = \frac{n-2}{n-1} I_{n-2}$, we get $I_5 = \frac{3}{4} I_3 = \frac{3}{4} \cdot \frac{1}{2} I_1$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{16} \pi. \quad]$$

29. Use the relation

$$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \dots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}},$$

$$\text{to prove that } \int_0^{\frac{\pi}{2}} \log \sin x \, dx = -\frac{\pi}{2} \log 2.$$

[Divide the interval $(0, \frac{\pi}{2})$ into n parts by the points $0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{(n-1)\pi}{2n}$.

Now $\sin \frac{\pi}{2n} \sin \frac{2\pi}{2n} \dots \sin \frac{(2n-1)\pi}{2n} = \frac{2n}{2^{2n-1}}$, whence

$$\sin \frac{\pi}{2n} \sin \frac{2\pi}{2n} \dots \sin \frac{(n-1)\pi}{2n} = \sqrt{\frac{2n}{2^{2n-1}}}.$$

$$\text{Now } \int_0^{\frac{\pi}{2}} \log \sin x \, dx$$

$$= \frac{\pi}{2n} \left\{ \log \sin \frac{\pi}{2n} + \log \sin \frac{2\pi}{2n} + \dots + \log \sin \frac{(n-1)\pi}{2n} \right\} = \text{etc.} \quad]$$

CHAPTER 13

EULERIAN INTEGRALS

13.1. The integral $\int_0^1 x^{n-1} (1-x)^{m-1} dx$.

Then $\int_0^1 x^{n-1} (1-x)^{m-1} dx$ can be immediately calculated when m and n are positive and one of them is an integer. Let m be an integer; then

$$\begin{aligned} \int_0^1 x^{n-1} (1-x)^{m-1} dx &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\ &= \frac{1 \cdot 2 \cdot 3 \cdots (m-1)}{n (n+1) \cdots (n+m-1)}. \end{aligned}$$

From the formula $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, we at once have

$$\int_0^1 x^{n-1} (1-x)^{m-1} dx = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$$

Now integrating by parts the integral $\int_0^1 x^{n-1} (1-x)^{m-1} dx$,

we get

$$\int x^{n-1} (1-x)^{m-1} dx = \frac{x^n}{n} (1-x)^{m-1} + \frac{m-1}{n} \int x^n (1-x)^{m-2} dx.$$

As n and $m-1$ are both positive, the term $x^n (1-x)^{m-1}$ vanishes for both limits;

$$\therefore \int_0^1 x^{n-1} (1-x)^{m-1} dx = \frac{m-1}{n} \int_0^1 x^n (1-x)^{m-2} dx.$$

By a repeated application of this formula, the integral can be made to depend on $\int_0^1 x^{m+n-2} dx$, the value of which is $\frac{1}{m+n-1}$.

Hence we have

$$\int_0^1 x^{n-1} (1-x)^{m-1} dx = \frac{1 \cdot 2 \cdot 3 \cdots (m-1)}{n(n+1) \cdots (n+m-1)}.$$

When n is an integer,

$$\begin{aligned} \int_0^1 x^{n-1} (1-x)^{m-1} dx &= \int_0^1 x^{n-1} (1-x)^{n-1} dx \\ &= \frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{n(n+1) \cdots (m+n-1)}. \end{aligned}$$

When neither m nor n is an integer, the above integral is one of the most important integrals in analysis, and is called the First Eulerian integral or the Beta Function.

13.2. First Eulerian Integral of the Beta Function.

Consider the integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$.

If $m \geq 1$ and $n \geq 1$, the integrand has no point of infinite discontinuity in $(0, 1)$ and therefore the integral exists as a proper integral.

If $m < 1$, $x=0$ is a point of infinite discontinuity and if $n < 1$, $x=1$ is a point of infinite discontinuity. Thus for $m < 1$, $n < 1$, the integral is an improper one. We write

$$\begin{aligned} \int_0^1 x^{m-1} (1-x)^{n-1} dx &= \int_0^a x^{m-1} (1-x)^{n-1} dx \\ &\quad + \int_a^1 x^{m-1} (1-x)^{n-1} dx \end{aligned}$$

$$= u_1 + u_2 \text{ say, where } 0 < a < 1.$$

We have to test u_1 for convergence at $x=0$ and u_2 for convergence at $x=1$.

Taking u_1 and writing $f(x)=x^{m-1}(1-x)^{n-1}=\frac{(1-x)^{n-1}}{x^{1-m}}$ and

$$\phi(x)=\frac{1}{x^{1-m}}, \text{ we have } \lim_{x \rightarrow 0} \frac{f(x)}{\phi(x)}=1.$$

As $\int_0^a \frac{dx}{x^{1-m}}$ is convergent when $1-m < 1$ or $m > 0$, we deduce that u_1 is convergent at $x=0$ if, and only if $m > 0$.

Next taking u_2 and writing $f(x)=\frac{x^{m-1}}{(1-x)^{1-n}}$ and $\phi(x)=\frac{1}{(1-x)^{1-n}}$,

we have $\lim_{x \rightarrow 1-0} \frac{f(x)}{\phi(x)}=1$. As $\int_a^1 \frac{dx}{(1-x)^{1-n}}$ is convergent when

$1-n < 1$ or $n > 0$, we find that u_2 is convergent at $x=1$ if, and only if, $n > 0$.

Thus we find that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ exists when $m > 0$ and

$n > 0$. So for positive values of m and n , the integral defines a function of m and n and is known as the **First Eulerian Integral** or the **Beta Function**.

We write $\int_0^1 x^{m-1}(1-x)^{n-1} dx \ (m > 0, n > 0) = B(m, n)$.

Obviously $B(m, n) = B(n, m)$.

Again putting $x = \sin^2 \theta$ in $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \ (m > 0,$

$n > 0)$, we see that

$$B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta ;$$

if $2m-1$ and $2n-1$ are positive integers, we have

$$B(m, n) = \frac{(2m-2)(2m-4)\cdots(2n-2)(2n-4)\cdots}{(2m+2n-2)(2m+2n-4)\cdots}$$

multiplied by $\frac{\pi}{I}$ if $2m-1$, $2n-1$ are both even positive integers.

13.3. Transformations of $B(m, n)$.

The following transformations of $B(m, n)$ are easily worked out.

(a) In $B(m, n) = \int_0^1 y^{m-1}(1-y)^{n-1} dy$, put $y = \frac{x}{a}$; then

$$B(m, n) = \frac{1}{a^{m+n-1}} \int_0^a x^{m-1}(a-x)^{n-1} dx.$$

(b) In the same put $y = \frac{x}{1+x}$, then

$$B(m, n) = \int_0^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1} dx}{(1+x)^{m+n}}.$$

(c) $B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx.$

In $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$, put $x = \frac{1}{y}$; then

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_1^0 \frac{-\frac{1}{y^2} dy}{y^{m-1} \left(1 + \frac{1}{y}\right)^{m+n}} = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy.$$

$$\therefore B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.$$

(d) In $B(m, n) = \int_0^1 y^{m-1}(1-y)^{n-1} dy$, put $\frac{y}{1+a} = \frac{x}{x+a}$; then

$$B(m, n) = (1+a)^m a^n \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}} dx.$$

Hence
$$\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{B(m, n)}{(1+a)^m a^n}.$$

(e) From (b), $B(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$. In this put $y = \frac{ax}{b}$, then

$$B(m, n) = a^m b^n \int_0^\infty \frac{x^{m-1} dx}{(ax+b)^{m+n}}.$$

(f) In $\int_0^1 y^{m-1} (1-y)^{n-1} dy$, put $\frac{y}{a+b} = \frac{x}{a+bx}$; then

$$B(m, n) = (a+b)^m a^n \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} dx.$$

$$\therefore \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{B(m, n)}{(a+b)^m a^n}.$$

(g) Taking $B(m, n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$ and transforming it by

putting $y = \frac{bx}{a}$, we get

$$B(m, n) = a^m b^n \int_0^\infty \frac{x^{n-1} dx}{(a+bx)^{m+n}}.$$

Next putting $x = \tan^2 \theta$, we have

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{2m-1} \theta \sin^{2n-1} \theta d\theta}{(a \cos^2 \theta + b \sin^2 \theta)^{m+n}} = \frac{B(m, n)}{2a^m b^n}.$$

(h) Putting $y = \frac{x-a}{b-a}$ in $B(m, n) = \int_0^1 y^{m-1} (1-y)^{n-1} dy$, we get,

$$B(m, n) = \frac{1}{(b-a)^{m+n-1}} \int_a^b (x-a)^{m-1} (b-x)^{n-1} dx.$$

Here the limits 0, 1 are changed into arbitrary numbers a, b ($a < b$).

13'4. Second Eulerian Integral or the Gamma Function.

Consider the integral

$$\int_0^{\infty} e^{-x} x^{n-1} dx.$$

If $n \geq 1$, the integrand is continuous at $x=0$ and if $n < 1$, the integrand has an infinite discontinuity at $x=0$.

We write

$$\int_0^{\infty} e^{-x} x^{n-1} dx = \int_0^1 e^{-x} x^{n-1} dx + \int_1^{\infty} e^{-x} x^{n-1} dx = u_1 + u_2 \text{ say}$$

We have to examine u_1 for convergence at $x=0$ and u_2 for convergence at ∞ .

Taking u_1 and writing $f(x) = e^{-x} x^{n-1} = \frac{e^{-x}}{x^{1-n}}$ and taking $\phi(x) = \frac{1}{x^{1-n}}$, we have

$$\lim_{x \rightarrow 0} \frac{f(x)}{\phi(x)} = 1,$$

and as $\int_0^1 \frac{dx}{x^{1-n}}$ is convergent at 0, if and only if $1-n < 1$.

i.e., $n > 0$, it follows that $u_1 = \int_0^1 e^{-x} x^{n-1} dx$ is convergent if

$n > 0$.

Next to establish the convergence of u_2 at ∞ .

We have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \cdots$$

Thus when $x > 0$, $e^x > \frac{x^r}{r!}$ (r any positive integer).

$$\therefore e^{-x} x^{n-1} < \frac{r!}{x^{r-n+1}}.$$

But whatever n may be, we can choose r so that $r-n+1 > 1$. Also we know that

$$\int_1^{\infty} \frac{dx}{x^{r-n+1}}$$

is convergent if $r-n+1 > 1$. It follows that

$$\int_1^{\infty} e^{-x} x^{n-1} dx < r! \int_1^{\infty} \frac{dx}{x^{r-n+1}}$$

is convergent whatever n may be.

$$\therefore \int_1^{\infty} e^{-x} x^{n-1} dx \text{ is convergent at } \infty.$$

Thus $\int_0^{\infty} e^{-x} x^{n-1} dx$ is convergent for all values of n greater than 0.

The function $\int_0^{\infty} e^{-x} x^{n-1} dx$ has thus a meaning for all positive values of n , and is known as the **Gamma Function**. We write

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx.$$

Cor. By integrating $\int_0^{\infty} e^{-x} x^n dx$ by parts, we easily get

$$\Gamma(n+1) = n \Gamma(n)$$

and if n be a positive integer, $\Gamma(n) = (n-1)!$

135. Relation between Beta and Gamma functions.

We have

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx.$$

[Art. 133 (b)]

In $\Gamma(m) = \int_0^{\infty} e^{-x} x^{m-1} dx$, put zx for x where z does not contain

x ; we get

$$\Gamma(m) = \int_0^{\infty} e^{-x} x^{m-1} dx = \int_0^{\infty} e^{-zx} z^m x^{m-1} dx;$$

$$\therefore \Gamma(m) e^{-z} z^{n-1} = \int_0^{\infty} e^{-z(1+x)} z^{m+n-1} x^{m-1} dx.$$

$$\begin{aligned} \text{So } \Gamma(m) \int_0^{\infty} e^{-z} z^{n-1} dz &= \int_0^{\infty} \int_0^{\infty} e^{-z(1+x)} z^{m+n-1} x^{m-1} dx dz, \\ &= \int_0^{\infty} \left[\int_0^{\infty} e^{-z(1+x)} z^{m+n-1} dz \right] x^{m-1} dx. \end{aligned}$$

Now putting $z(1+x) = y$, we have

$$\int_0^{\infty} e^{-z(1+x)} z^{m+n-1} dz = \frac{1}{(1+x)^{m+n}} \int_0^{\infty} e^{-y} y^{m+n-1} dy = \frac{\Gamma(m+n)}{(1+x)^{m+n}}.$$

$$\text{Hence } \Gamma(m) \int_0^{\infty} e^{-z} z^{n-1} dz = \int_0^{\infty} \frac{\Gamma(m+n) x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \Gamma(m+n) \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx,$$

$$\text{or, } \Gamma(m)\Gamma(n) = \Gamma(m+n).B(m, n),$$

$$\text{i.e., } B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

$$\text{Cor. 1. Since } B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta, \quad (\text{Art. 13'2})$$

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta.$$

$$\text{Cor. 2. As } \Gamma(1) = 1, \Gamma(n)\Gamma(1-n) = \int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}.$$

(Ex. 27, Ch. 12)

13'6. Duplication formula : (Legendre).

We have

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta. \quad \dots (1)$$

$$m=n=\frac{1}{2} \text{ in (1) gives } \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = 2 \int_0^{\frac{\pi}{2}} d\theta = \pi, \therefore \Gamma(\frac{1}{2}) = \sqrt{\pi}. \quad \dots (2)$$

$m=n$ in (1) gives

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2m-1}\theta d\theta = \frac{1}{2^{2m-2}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} 2\theta d\theta,$$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1}\phi d\phi \quad (2\theta = \phi).$$

$$\begin{aligned}\text{Now } \int_{\frac{\pi}{2}}^{\pi} \sin^{2m-1} \phi \, d\phi &= \int_0^{\frac{\pi}{2}} \cos^{2m-1} \psi \, d\psi \quad (\text{by putting } \phi = \frac{\pi}{2} + \psi) \\ &= \int_0^{\frac{\pi}{2}} \sin^{2m-1} \psi \, d\psi.\end{aligned}$$

$$\therefore \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-2}} \int_0^{\frac{\pi}{2}} \sin^{2m-1} \phi \, d\phi. \quad \dots (3)$$

In (1) taking $n = \frac{1}{2}$;

$$\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \, d\theta. \quad \dots (4)$$

From (2), (3), (4),

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\sqrt{\pi}}{\Gamma(m+\frac{1}{2})},$$

$$\text{or, } \sqrt{\pi} \Gamma(2m) = 2^{2m-1} \Gamma(m)\Gamma(m+\frac{1}{2}).$$

Examples :

$$1. \text{ Show that } \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right) \quad \text{where}$$

$$m > -1, n > -1$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n}{2} + 1\right)}.$$

Deduce the values of

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta, \quad \int_0^{\frac{\pi}{2}} \sin^m \theta \, d\theta$$

when m and n are positive integers.

2. Show that

$$(i) B(m, n) = B(m+1, n) + B(m, n+1).$$

$$(ii) \quad \frac{B(m+1, n)}{m} = \frac{B(m, n+1)}{n} = \frac{B(m, n)}{m+n}.$$

[For (i), use the identity : $x^{m-1}(1-x)^{n-1} \equiv x^{m-1}(1-x)^{n-1}(x+1-x).$

For (ii) integrate, $\int_0^1 x^m(1-x)^{n-1} dx$ by parts and use (i).]

3. Prove that

$$\int_0^1 x^{m-1}(1-x^p)^{n-1} dx = \frac{1}{p} B\left(\frac{m}{p}, n\right).$$

[Put $x^p = z.$]

4. Show that if a, b, m, n be all positive,

$$\int_0^a x^{m-1}(a^b - x^b)^{n-1} dx = \frac{a^{b(n-1)+m}}{b} B\left(\frac{m}{b}, n\right).$$

$$\left[\int_0^a x^{m-1}(a^b - x^b)^{n-1} dx = a^{bn-b} \int_0^1 x^{m-1} \left\{ 1 - \left(\frac{x}{a}\right)^b \right\}^{n-1} dx ; \right.$$

now put $\left(\frac{x}{a}\right)^b = z.$]

5. Deduce from Ex. 4 or obtain directly the values of

$$(i) \quad \int_0^1 \frac{x^2 dx}{\sqrt{1-x^3}} ; \quad (ii) \quad \int_0^2 x^3(16-x^4)^{-\frac{1}{4}} dx. \left[\frac{2}{3}, \frac{8}{3} \right]$$

6. Show that

$$\int_0^1 \left(1 - \frac{x}{n}\right)^n x^{m-1} dx = n^m B(m, n+1) = \frac{n! n^m}{m(m+1)\cdots(m+n)},$$

where $m > 0$ and n a positive integer.

[Use Ex. 4 to get the first part and Art. 13'1 to get the second part.]

$$7. \text{ Prove that } \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta d\theta}{(a \sin^2\theta + b \cos^2\theta)^{m+n}} = \frac{B(m, n)}{2a^m b^n}$$

$$= \frac{\Gamma(m)\Gamma(n)}{2a^m b^n \Gamma(m+n)}. \quad (m > 0, n > 0)$$

$$\left[\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx. \right]$$

Put $x = \frac{ay}{b}$ where $a > 0, b > 0$; then

$$\frac{\Gamma(m)\Gamma(n)}{a^m b^n \Gamma(m+n)} = \int_0^{\infty} \frac{y^{m-1} dy}{(ay+b)^{m+n}}.$$

Now put $y = \tan^2\theta$ to get the required result.]

$$8. \text{ Evaluate } \int_0^{\infty} e^{-x^2} dx. \quad (C. H. 1962, 1968)$$

$$\left[\text{Putting } x^2 = z, \int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \int_0^{\infty} e^{-z} z^{-\frac{1}{2}} dz \right.$$

$$\left. = \frac{1}{2} \int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz = \frac{\Gamma(\frac{1}{2})}{2} = \frac{\sqrt{\pi}}{2}. \right]$$

9. Discuss the convergence or divergence of

$$\int_0^1 x^{m-1} (\log x)^n dx. \quad (C. H. 1963)$$

$$\left[\text{Put } x = e^{-z}; \int_0^1 x^{m-1} (\log x)^n dx \text{ is transformed into} \right.$$

$$\int_0^{\infty} (-1)^n \cdot e^{-mz} z^n dz = (-1)^n \int_0^{\infty} e^{-mz} z^n dz; \quad m \text{ must be } > 0 \text{ so that}$$

the index of e in the integrand may be negative. On putting $mz = y$,

$$\text{the integral is changed into } (-1)^n \int_0^{\infty} \frac{e^{-y} y^n}{m^{n+1}} dy = (-1)^n \frac{\Gamma(n+1)}{m^{n+1}}$$

provided $n+1 > 0$. Thus the integral converges when $m > 0$, $n > -1$.]

10. Show that

$$\int_1^{\infty} x^{m-1} (\log x)^n dx$$

converges if $m < 0$ and $n > -1$.

[Put $x = e^z$.]

11. Show that $\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}}$.

12. Prove that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(\sin x)}} \times \int_0^{\frac{\pi}{2}} \sqrt{(\sin x)} dx = \pi.$$

13. Show that

$$\int_0^1 \frac{x^2 dx}{(1-x^4)^{\frac{1}{2}}} \times \int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \frac{\pi}{4\sqrt{2}}.$$

14. Show that $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}} = \frac{\pi}{3}$.

15. Show that

(i) $B(m, m) = 2^{1-2m} B(m, \frac{1}{2}), (m > 0).$

(ii) $B(m, m) \cdot B(m + \frac{1}{2}, m + \frac{1}{2}) = \frac{\pi}{m} \cdot 2^{1-4m}, (m > 0).$

16. Prove that

$$\int_0^1 \frac{dx}{\sqrt[2n]{1-x^n}} = \frac{\sqrt[2n]{\pi}}{n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)} = \frac{2^{\frac{2}{n}-1} \left[\Gamma\left(\frac{1}{n}\right) \right]^2}{n \Gamma\left(\frac{2}{n}\right)}.$$

[The first part of the result is easily obtained by putting $x^n = z$.
For the second part, use the duplication formula.]

17. Show that

$$\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}} = \frac{\frac{\pi}{n}}{\sin \frac{\pi}{n}}$$

CHAPTER 14

POWER SERIES

14.1. Some Basic Concepts*.

(a) Convergence of Infinite Series.

The infinite series $u_1 + u_2 + u_3 + \dots$ is convergent if

Lt $S_n = S$ where $S_n = u_1 + u_2 + \dots + u_n$; this may be stated $n \rightarrow \infty$

as $|S_n - S| < \epsilon$ for $n \geq m$,

m depending on ϵ .

Now $|S_n - S| = |u_{n+1} + u_{n+2} + \dots| = |R_n|$ say, the above condition is then

$$|R_n| < \epsilon \text{ for } n \geq m$$

m depending on ϵ .

(b) Absolute Convergence.

If the infinite series be $u_1 + u_2 + u_3 + \dots$ and if $|u_1| + |u_2| + |u_3| + \dots$ is convergent, the series is said to be *absolutely* convergent.

We have the theorem: An absolutely convergent series is convergent.

If Σu_n be absolutely convergent, by definition $\Sigma |u_n|$ is convergent.

Now, $u_n + |u_n| = 2|u_n|$ or 0, according as $u_n \geq 0$. Therefore every term of $\Sigma(u_n + |u_n|) \geq 0$ and $\leq$ the corresponding term of $\Sigma(|u_n| + |u_n|)$ i.e., of $2\Sigma|u_n|$ which is convergent.

So $\Sigma(u_n + |u_n|)$ is convergent, i.e., $\Sigma u_n + \Sigma|u_n|$ is convergent.

It follows that Σu_n is convergent.

(c) Convergence of Series whose terms are functions of x .

A series of the form $\Sigma u_n(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots$ whose terms are functions of x is said to be convergent for the

* Chapter 21 may be consulted at this stage.

value x and to have $S(x)$ for its sum, if this value of x having been first inserted in the different terms of the series, and any positive number ϵ having been chosen, as small as we please, there is a positive integer m such that

$$|S(x) - S_n(x)| < \epsilon \text{ when } n \geq m.$$

Here m depends upon ϵ and the particular value of x chosen.

(d) Uniform Convergence.

Let the series $u_1(x) + u_2(x) + u_3(x) + \dots$ converge for all values of x in $a \leq x \leq b$. It is said to converge *uniformly* in the interval, if any positive number ϵ having been chosen, however small, there is a positive integer m such that for all values of x in (a, b) ,

$$\begin{aligned} |S(x) - S_n(x)| &< \epsilon \text{ when } n \geq m, \\ \text{i.e., } |R_n(x)| &< \epsilon \quad \text{when } n \geq m. \end{aligned}$$

Here m depends on ϵ only and not on x .

Thus for uniform convergence, any arbitrary positive number ϵ being chosen, the same value of m is to serve for all the values of x in the interval.

It is therefore true that if $\sum u_n(x)$ converges uniformly,

$$|R_n(x)| < \epsilon$$

for each value of x in (a, b) when $n \geq m$.

14.2. Power Series.

A series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots, \quad (1)$$

in which the a 's are real numbers for a real series and the exponents of x are positive integer, is called a **power series**.

A more general form of a power series is

$$\begin{aligned} \sum a_n (x - x_0)^n &= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots \\ &\quad + a_n (x - x_0)^n + \dots \end{aligned} \quad \dots (2)$$

The form (2) can be reduced to the form (1) by putting $x - x_0 = z$ and so we need consider only the form (1) for a discussion of the power series.

14'3. Three types of Power Series.

Power series can be divided into three classes according to the nature of their convergence. It should however be noted that there is one value of x , viz., $x=0$ for which every power series is convergent.

(a) Power series convergent for all values of x . As an example we consider the series : $1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\dots$ Here

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{x}{n}$$

and whatever be the value of x , the ratio becomes < 1 if n be sufficiently large. Hence the series converges for all values of x .

(b) Power series convergent for no value of x except $x=0$.

As an example we have the series : $1+x+2!x^2+\dots+n!x^n+\dots$
Here

$$\left| \frac{u_{n+1}}{u_n} \right| = nx$$

and if $x \neq 0$, this ratio becomes > 1 for large n , showing that the series diverges for all values of x except $x=0$.

(c) Power series convergent for some values of x and divergent for other values.

An example is : $1+x+x^2+\dots+x^n+\dots$

Neglecting the first term $u^n = x$; and so the series converges for $-1 < x < 1$ and diverges for $|x| > 1$; it also diverges for $x = \pm 1$.

14'4. Determination of the region of convergence of a power series.

The values of x for which a power series converges can be easily determined by the ratio test if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ exists.

If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} \neq 0$, the power series

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

converges absolutely, and therefore converges, for all values of x if $|x| < R$ and diverges if $|x| > R$.

Neglecting the first term and using the ratio test, we have

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \cdot x \right| = \frac{|x|}{R}.$$

Hence the series converges absolutely if $\frac{|x|}{R} < 1$ or $|x| < R$

and diverges if $\frac{|x|}{R} > 1$ or $|x| > R$.

Nothing is proved in this as to what happens when $x = \pm R$.

If $R = \infty$, i.e., $\frac{1}{R} = 0$, $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \cdot x \right| = \frac{|x|}{R} = 0$; and the series converges for all values of x .

If $R = 0$, or $\frac{1}{R} = \infty$, $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \infty$ and the series diverges for all values of x , except $x = 0$.

14.5. Theorems on Power Series,

(a) **Theorem I.** If the power series

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

converges for a particular value x_0 of x , it converges absolutely for all values of x such that $|x| < |x_0|$. (Abel).

Since the series converges for $x = x_0$, we must have

$|a_n x_0^n| < \epsilon$ and so $|a_n x_0^n| < M$ for all n , where M is a definite finite number.

Now $|x| < |x_0|$, so $\left| \frac{x}{x_0} \right| < 1$.

Hence $|a_0| + |a_1x| + |a_2x^2| + \dots + |a_nx^n| + \dots$

$$= |a_0| + |a_1x_0| \left| \frac{x}{x_0} \right| + |a_2x_0^2| \left| \frac{x}{x_0} \right|^2 + \dots$$

$$+ |a_nx_0^n| \left| \frac{x}{x_0} \right|^n + \dots$$

$$< M \left(1 + \left| \frac{x}{x_0} \right| + \left| \frac{x}{x_0} \right|^2 + \dots + \left| \frac{x}{x_0} \right|^n + \dots \right)$$

which is convergent. Hence the theorem.

(b) **Theorem II.** If a power series diverges for $|x| = x_0$, then it diverges for all values of x for which $|x| > |x_0|$.

For if it converges when $x=x_1$ such that $|x_1| > |x_0|$, it must converge absolutely when $x=x_0$ (as $|x_0| < |x_1|$) by theorem I, and this contradicts the hypothesis.

Remark: These theorems show that the region of convergence cannot include a point where the series diverges, nor can the region of divergence include a point where the series converges. The greatest possible value R of x_0 is called the radius of convergence and the interval $(-R, R)$ is called the interval of convergence. We shall now prove the existence of the number R .

Suppose that there is a value of x other than zero for which $\Sigma a_n x^n$ converges and that there is a value x' of x for which it is non-convergent.

Choose a positive number a greater than $|x'|$; then by what has been proved above, the series is non-convergent for $x=a$.

Divide the real numbers in $(0, a)$ into two classes. The lower class is to contain every real number r such that $\Sigma a_n x^n$ is convergent if $|x|=r$. The upper class is to contain every real number r' such that $\Sigma a_n x^n$ does not converge if $|x|=r'$. Both classes exist and every r is less than any r' .

This division defines a real positive number R which separates the classes and may belong to either class. The number R is such that $\Sigma a_n x^n$ converges absolutely if $|x| < R$ and is non-convergent if $|x| > R$. Nothing is proved about what happens when $|x| = R$.

For a complex series $\Sigma a_n z^n$, the circle with centre at the origin and radius R is called the **circle of convergence** and R is the **radius of convergence**. The series converges absolutely or is non-convergent according as the point z is inside or outside the circle. If z is on the circumference, the character of the series is not determined.

(c) **Theorem III.** If R be the radius of convergence of a power series $\Sigma a_n x^n$, then the series is absolutely and uniformly convergent in $(-r, r)$, where r is any number less than R .

In other words—A power series is absolutely and uniformly convergent in a range strictly within the interval of convergence $(-R, R)$.

Let x_0 be a number such that $r < x_0 < R$, then by theorem I, $\sum a_n x_0^n$ is absolutely convergent.

$$\therefore |a_n x_0^n| + |a_{n+1} x_0^{n+1}| + \dots < \epsilon \text{ when } n \geq m.$$

And if x be any point in $(-r, r)$, i.e., if $-r \leq x \leq r$,

$$\begin{aligned} & |a_n x^n| + |a_{n+1} x^{n+1}| + \dots \\ & < |a_n x_0^n| + |a_{n+1} x_0^{n+1}| + \dots \\ & < \epsilon \end{aligned}$$

when $n \geq m$ and for every x such that $-r \leq x \leq r$. It follows that the series is *uniformly* convergent in the closed interval $(-r, r)$; that it is absolutely convergent in $(-r, r)$ follows directly from theorem I.

(d) **Theorem IV.** A power series represents a function which is continuous within the interval of convergence.

In other words—For a power series, the sum-function is continuous within the interval of convergence.

We have seen that a power series is a series of continuous functions which converges uniformly within the interval of convergence (theorem III).

$$\text{Let } S(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_0^{\infty} a_n x^n, \quad \dots \quad \dots (1)$$

$$S_n(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}, \quad \dots \quad \dots (2)$$

$$R_n(x) = a_n x^n + a_{n+1} x^{n+1} + \dots \quad \dots \quad \dots (3)$$

Since (1) converges uniformly, we know that however small the positive number ϵ may be, there is a positive integer m such that

$$|S(x) - S_n(x)| < \frac{\epsilon}{3}, \text{ when } n \geq m,$$

the same m serving for all values of x within the interval of convergence. Choosing such a value of n ,

$$S(x) = S_n(x) + R_n(x)$$

where $|R_n(x)| < \frac{\epsilon}{3}$ for all x within the interval of convergence.

$S_n(x)$, being the sum of a finite number n of continuous functions, is continuous within the interval. Thus there is a positive number η such that x' and x being any two values of x within the interval,

$$|S_n(x') - S_n(x)| < \frac{\epsilon}{3} \text{ whenever } |x' - x| \leq \eta.$$

Now $S(x') = S_n(x') + R_n(x')$ where $|R_n(x')| < \frac{\epsilon}{3}$.

And $S(x') - S(x) = S_n(x') - S_n(x) + R_n(x') - R_n(x)$;

$$\begin{aligned} \therefore |S(x') - S(x)| &\leq |S_n(x') - S_n(x)| + |R_n(x')| + |R_n(x)|, \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}, \\ &< \epsilon. \end{aligned}$$

Thus $S(x)$ is continuous within the interval of convergence.

Note. The following examples will show that a series whose terms are continuous functions of x need not necessarily have a continuous sum.

(1) Consider the series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n} + \cdots$$

When $x \neq 0$, it is a geometrical series with common ratio $\frac{1}{1+x^2} < 1$. So the series is convergent and its sum

$$S(x) = \frac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2 (x \neq 0).$$

Hence $\lim_{x \rightarrow 0} S(x) = \lim_{x \rightarrow 0} (1 + x^2) = 1$. But $S(0) = 0$, as every

term is 0 when $x = 0$.

Thus $S(x)$ approaches a definite limit 1 when $x \rightarrow 0$, but this limit is different from the value of the function for $x = 0$, which means that $S(x)$ is not continuous at $x = 0$.

(2) Again consider the series

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \cdots, \quad (x \geq 0).$$

Here $u_n(x) = \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$, and $S_n(x) = 1 - \frac{1}{nx+1}$.

Thus when $x > 0$, $\lim_{n \rightarrow \infty} S_n(x) = S(x) = 1$.

When $x = 0$, $\lim_{n \rightarrow \infty} S_n(x) = 0$, since $S_n(0) = 0$.

Thus $S(x)$ is discontinuous at $x = 0$.

The sum-function is continuous in an interval only if the series is uniformly convergent therein.

(e) **Theorem V.** A power series can be integrated term by term within the interval of convergence.

We have $S(x) = S_n(x) + R_n(x)$.

Since a power series is uniformly convergent within the interval of convergence, for any x within the interval

$$|R_n(x)| < \epsilon \text{ for } n \geq n.$$

If now c_1 and c_2 ($c_1 < c_2$) be two points within the interval,

$$\left| \int_{c_1}^{c_2} R_n(x) dx \right| \leq \int_{c_1}^{c_2} |R_n(x)| dx < \int_{c_1}^{c_2} \epsilon dx = \epsilon (c_2 - c_1).$$

$$\text{Now } \int_{c_1}^{c_2} S(x) dx = \int_{c_1}^{c_2} S_n(x) dx + \int_{c_1}^{c_2} R_n(x) dx.$$

$$\therefore \left| \int_{c_1}^{c_2} S(x) dx - \int_{c_1}^{c_2} S_n(x) dx \right| = \left| \int_{c_1}^{c_2} R_n(x) dx \right|$$

$$\leq \int_{c_1}^{c_2} |R_n(x)| dx < \epsilon (c_2 - c_1).$$

$$\text{So } \int_{c_1}^{c_2} S(x) dx = \lim_{n \rightarrow \infty} \int_{c_1}^{c_2} S_n(x) dx$$

$$= \int_{c_1}^{c_2} a_0 dx + \int_{c_1}^{c_2} a_1 x dx + \int_{c_1}^{c_2} a_2 x^2 dx + \dots$$

(f) **Theorem VI.** A power series can be differentiated term by term within the interval of convergence.

$$\text{Let } S(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots, \quad \dots (1)$$

$$\text{and } \phi(x) = a_1 + 2a_2 x + \dots + (n+1)a_{n+1} x^n + \dots, \quad \dots (2)$$

obtained by differentiating $S(x)$ term by term. As (1) and (2) are both power series, they are both uniformly convergent within their respective intervals of convergence.

We shall first prove that (2) has the same interval of convergence as (1). If R be the radius of convergence of (1), then

$$\text{Lt} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} \quad (\text{Art. 14'4}).$$

For the series (2),

$$\text{Lt} \left| \frac{u_{n+1}}{u_n} \right| = \text{Lt} \left| \frac{(n+1)a_{n+1}}{n a_n} \right| = \text{Lt} \left| 1 + \frac{1}{n} \right| \cdot \text{Lt} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}.$$

Hence the first derived series has the same interval of convergence as the original series.

Now $\phi(x)$ being a power series it can be integrated term by term within the interval of convergence $(-R, R)$ (Theorem V). If x be a point in this interval

$$\int_0^x \phi(x) dx = \Sigma \int_0^x (n+1)a_{n+1}x^n dx = \Sigma a_{n+1}x^{n+1} = S(x).$$

Again $\phi(x)$ being the sum of the power series (2) is a continuous function of x in $(-R, R)$ and by a property of definite integrals,

$$S'(x) = \phi(x) = a_1 + 2a_2x + 3a_3x^2 + \dots + (n+1)a_{n+1}x^n + \dots$$

which proves the theorem.

Cor. A power series can be integrated or differentiated term by term within the interval convergence as often as we please.

(g) **Theorem VII.** Uniqueness of Identity Theorem for power series :

There cannot be two different power series with the same interval of convergence and having the same sum in the interval.

If possible let $\Sigma a_n x^n$ and $\Sigma b_n x^n$ be two power series satisfying the given condition in $(-R, R)$; then

$$a_0 + a_1x + a_2x^2 + \dots = b_0 + b_1x + b_2x^2 + \dots$$

Differentiating successively both sides term by term we have by Theorem VI

$$\begin{aligned} a_1 + 2a_2x + 3a_3x^2 + \dots &= b_1 + 2b_2x + 3b_3x^2 + \dots \\ 1.2a_2 + 2.3a_3x + 3.4a_4x^2 + \dots &= 1.2b_2 + 2.3b_3x + 3.4b_4x^2 + \dots \\ \dots & \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

Putting $x=0$ in these relations, we get successively

$$a_0 = b_0, a_1 = b_1, a_2 = b_2, \dots, a_n = b_n, \dots$$

proving the identity of the two series,

(h) **Theorem VIII.** We state without proof the following properties of power series.

1. Two power series as long as they converge may be added and subtracted term by term.

$$\Sigma a_n x^n \pm \Sigma b_n x^n = \Sigma (a_n \pm b_n) x^n.$$

2. Two power series can be multiplied term by term for values of x within the common portion of their intervals of convergence. For such values of x ,

$$\sum_0^{\infty} a_n x^n \cdot \sum_0^{\infty} b_n x^n = \sum_0^{\infty} (a_0 b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0) x^n.$$

3. We may divide by a power series of positive radius if the constant term is $\neq 0$ and provided we restrict ourselves to small values of x (Knopp).

14.6. We state without proof the following theorem due to Abel.

If a power series converges for either of the ends of the interval of convergence, the series is uniformly convergent at the point and the continuity of the sum of the series extends up to and includes the point.

Examples :

1. From the expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$ ($|x| < 1$), obtain the expansion of $(1+x)^{-3}$. What is its interval of convergence? (C. H. 1965)

2. From the same expansion, obtain by integration, the series for $\log(1+x)$, viz.,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (-1 < x \leq 1).$$

and deduce

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

[As $1 - x + x^2 - x^3 + \dots$ is absolutely and uniformly convergent when $|x| < 1$, by integrating between the limits 0 and x ($|x| < 1$), we get $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$.

This series is convergent when $x=1$ and non-convergent for

$x = -1$, and so by Abel's theorem (Art. 14'6), the interval of convergence extends up to and includes 1. Hence,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots (-1 < x \leq 1).$$

Putting $x=1$, we get

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.]$$

3. Obtain, by integration, from the identity

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^{n-1} t^{n-1} + (-1)^n \frac{t^n}{1+t},$$

the infinite series for $\log(1+x)$ in $(-1, 1)$.

$$[\text{Hint. } \int_0^x \frac{t^n}{1+t} dt < \int_0^x t^n dt = \frac{x^{n+1}}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty.]$$

4. Find the power series for $\tan^{-1}x$ and deduce the sum of

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

5. Show that $\int_0^1 \frac{\log(1+x)}{x} dx = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$

6. Find the power series development of $\int_0^x \frac{\sin t}{t} dt$ in the

neighbourhood of zero.

(C. H. 1960)

$$\left[x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \cdots \right]$$

7. From the equation

$$\sin^{-1}x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$$

obtain the power series for $\sin^{-1}x$.

(C. H. 1986)

$$\left[x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} \cdots \text{for } |x| \leq 1. \right]$$

8. Define the radius of convergence of a power series and explain

why it is so called. Obtain an expression for the radius of convergence of the power series

$$a_0 + a_1 z + a_2 z^2 + \dots$$

Prove that if $f(x) = \sum_0^{\infty} a_n x^n$ for $|x| < 1$, then

$$f'(x) = \sum_1^{\infty} n a_n x^{n-1} \text{ for } |x| < 1.$$

Show by an example that such differentiation is not always permissible for series in general. (C. H. 1963)

[Consider the series for which $S_n(x) = \frac{nx}{1+n^2 x^2}$ ($x \geq 0$);

then $S_n(0) = 0$, and also $S(x) = 0$; so that $S'(x) = 0$ ($x \geq 0$).

$$\text{Now } S'_n(0) = \lim_{h \rightarrow 0} \frac{S_n(h) - S_n(0)}{h} = \lim_{h \rightarrow 0} \frac{n}{1+n^2 h^2} = n.$$

$$\text{Thus } \lim_{n \rightarrow \infty} S'_n(0) = \infty, \text{ though } S'(0) = 0.$$

Hence the relation

$$\frac{d}{dx} \left[\lim_{n \rightarrow \infty} S_n(x) \right] = \lim_{n \rightarrow \infty} \frac{d}{dx} S_n(x)$$

is not always true.]

9. Prove that under suitable conditions two power series may be multiplied together and the product expressed as a power series.

If $f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$, prove by multiplication of the power series that

$$f(z) \times f(z') = f(z+z'). \quad (\text{C. H. 1962})$$

10. Find the radius of convergence of $z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$ and discuss

the case where z is on the circle of convergence.

[The radius of convergence is 1 so that the circle of convergence is the unit circle. When $|z| = 1$, the series converges; therefore when z is on the circle of convergence, the series converges absolutely and uniformly except when the amplitude of z is $\pm \pi$.]

11. Show that if $m > 0$,

$$\int_0^1 \frac{x^{m-1}}{1-x} dx = \frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+2} - \frac{1}{m+3} + \dots$$

Hence deduce the value of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

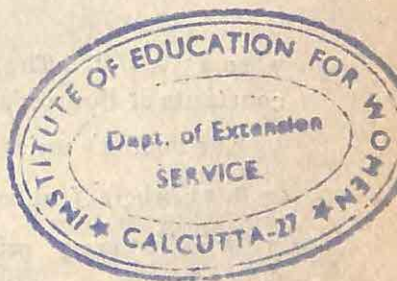
[Follow Ex. 27, Chapter 12.]

12. If $m > 0$, $n > 0$, show that

$$\int_0^1 \frac{x^{m-1}}{1+x^n} dx = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \dots$$

Deduce that

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \dots = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} + \log 2 \right).$$



CHAPTER 15

FOURIER SERIES

15.1. Fourier's Series and Constants.

Let $f(x)$ be an arbitrary function defined in the interval $(-\pi, \pi)$. If bounded, let it be integrable in this interval. If unbounded, let the improper integral $\int_{-\pi}^{\pi} f(t) dt$ be absolutely convergent. It can

be proved that then $\int_{-\pi}^{\pi} f(t) \cos nt dt$ and $\int_{-\pi}^{\pi} f(t) \sin nt dt$ exist for all values of n .

The trigonometrical series

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

$$\text{or, } \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \dots \quad \dots (1)$$

is called the **Fourier's series** for $f(x)$ when the constants $a_0, a_1, b_1, \dots$ are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt,$$

when $n=0, 1, 2, \dots$. The constants $a_0, a_1, b_1, \dots$ are called **Fourier's constants** or **Fourier's coefficients** for the function $f(x)$.

The Fourier's series for $f(x)$ converges to $f(x)$ at every point in $(-\pi, \pi)$ [where $f(x)$ is continuous; it converges to $\frac{1}{2} [f(x+0) + f(x-0)]$ at every point of ordinary discontinuity in the interval and to $\frac{1}{2} [f(-\pi+0) + f(\pi-0)]$ at $x=\pm\pi$, provided these limits exist.

Thus if $S(x)$ be the sum-function of the series,

$S(x) = f(x)$ at a point in $(-\pi, \pi)$ where $f(x)$ is continuous ;

$S(x) = \frac{1}{2} [f(x+0) + f(x-0)]$ at a point of ordinary discontinuity if the limits exist ;

$S(x) = \frac{1}{2} [f(-\pi+0) + f(\pi-0)]$ at $x = \pm\pi$, provided

$f(-\pi+0)$ and $f(\pi-0)$ exist ; and as the series is periodic with a period 2π ,

$$S(x+2m\pi) = S(x).$$

The last relation enables us to determine the value of the sum-function at a point outside the interval $(-\pi, \pi)$.

15.2. A rough verification of the above results. (Euler)

We assume that it is possible to expand the arbitrary function $f(x)$ in a trigonometrical series of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) ;$$

we assume further that the series may be integrated term by term after multiplying both sides by $\cos nx$ or $\sin nx$, ($n=0, 1, 2, 3, \dots$)

Bearing in mind that if m and n be positive integers,

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0,$$

$$\text{and } \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} \pi, & (m=n) \\ 0, & (m \neq n) \end{cases}$$

we have

$$\int_{-\pi}^{\pi} f(t) \cos nt \, dt = \pi a_n, \text{ or } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$$

$$\text{and } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt ; \text{ and similarly } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$

$$\text{Thus } f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \frac{1}{\pi} \sum_1^{\infty} \left\{ \cos nx \int_{-\pi}^{\pi} f(t) \cos nt dt \right. \\ \left. + \sin nx \int_{-\pi}^{\pi} f(t) \sin nt dt \right\} (-\pi \leq x \leq \pi).$$

Observation. In order that $f(x)$ may be equal to the corresponding series, we have to assume that

- (i) the trigonometric series should be convergent ;
- (ii) it should admit of term by term integration, a sufficient condition for which is that the convergence should be uniform ;
- (iii) the series being trigonometric is periodic in x with the period 2π ; hence unless $f(x)$ is itself periodic, the equality of $f(x)$ with the series will be confined only to the interval $(-\pi, \pi)$.

15.3. Special Cases : Cosine and Sine Series.

(a) **Cosine Series**—If the arbitrary function $f(x)$ given in $(-\pi, \pi)$ is an *even* function ; i.e., if $f(x) = f(-x)$ when $0 < x < \pi$, the Fourier's series for $f(x)$ becomes the Cosine Series :

$$\frac{1}{\pi} \int_0^{\pi} f(t) dt + \frac{2}{\pi} \sum_1^{\infty} \cos nx \int_0^{\pi} f(t) \cos nt dt, (0 \leq x \leq \pi).$$

For in this case

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \text{ leads to } a_0 = \frac{2}{\pi} \int_0^{\pi} f(t) dt,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt \text{ leads to } a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt dt,$$

$$\text{and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt \text{ leads to } b_n = 0.$$

(b) **Sine Series**—If the arbitrary function $f(x)$ in $(-\pi, \pi)$ be an *odd* function, i.e., $f(x) = -f(-x)$ when $0 < x < \pi$, the Fourier's series for $f(x)$ becomes the Sine Series :

$$\frac{2}{\pi} \sum_1^{\infty} \sin nx \int_0^{\pi} f(t) \sin nt \, dt, (0 \leq x \leq \pi).$$

In this case

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \text{ leads to } b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt,$$

and $a_n = 0$ for $n = 0, 1, 2, \dots$

15.4. Dirichlet's conditions.

A function $f(x)$ defined in the interval (a, b) will be said to satisfy Dirichlet's conditions in the interval when it satisfies one of the following conditions :

(i) $f(x)$ is bounded in (a, b) and the interval can be broken up into a finite number of open partial intervals in each of which $f(x)$ is monotonic.

(ii) $f(x)$ has a finite number of infinite discontinuities in the interval, but when arbitrarily small neighbourhoods of these points are excluded, $f(x)$ is bounded in the rest of the interval, and this can be broken up into a finite number of open partial intervals in each of which $f(x)$ is monotonic. Further, the improper integral

$$\int_a^b f(x) \, dx \text{ is to be absolutely convergent.}$$

It can be proved that when the arbitrary function $f(x)$ satisfies Dirichlet's conditions in the interval $(-\pi, \pi)$ and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt, (n = 0, 1, 2, \dots),$$

the Fourier's series

$$\frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges to $\frac{1}{2} [f(x+0) + f(x-0)]$ at every point in $-\pi < x < \pi$

where $f(x+0)$ and $f(x-0)$ exist; at $x=\pm\pi$, it converges to $\frac{1}{2} [f(-\pi+0)+f(\pi-0)]$ when $f(-\pi+0)$ and $f(\pi-0)$ exist.

15.5. Half-range series as a Cosine Series.

Let $f(x)$ be given in the interval $(0, \pi)$ and satisfy Dirichlet's conditions in that interval. Define $f(x)$ in $-\pi \leq x < 0$ by the equation $f(-x)=f(x)$. The function thus defined for $(-\pi, \pi)$ satisfies Dirichlet's conditions in this interval. On expanding $f(x)$ in a Fourier's series, it would be seen that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \text{ leads to } a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt,$$

$$\text{and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \text{ leads to } b_n = 0, (n=1, 2, \dots).$$

Hence the Fourier's series for $f(x)$ takes the form

$$\frac{1}{\pi} \int_0^{\pi} f(t) \, dt + \frac{2}{\pi} \sum_1^{\infty} \cos nx \int_0^{\pi} f(t) \cos nt \, dt, (0 \leq x \leq \pi).$$

Also from the way in which $f(x)$ is defined in $-\pi \leq x < 0$, we have $\frac{1}{2} [f(+0)+f(-0)] = f(+0)$ and $\frac{1}{2} [f(-\pi+0)+f(\pi-0)] = f(\pi-0)$, provided the limits $f(+0)$ and $f(\pi-0)$ exist.

It follows that when $f(x)$ is an arbitrary function satisfying Dirichlet's conditions in $(0, \pi)$, the sum of the half-range cosine series is equal to $\frac{1}{2} [f(x+0)+f(x-0)]$ at every point between 0 and π where $f(x+0)$ and $f(x-0)$ exist; and when $f(+0)$ and $f(\pi-0)$ exist, the sum is $f(+0)$ at $x=0$ and $f(\pi-0)$ at $x=\pi$.

15.6. Half-range series as a Sine Series.

Let $f(x)$ be given in the interval $(0, \pi)$ and satisfy Dirichlet's conditions in that interval. Define $f(x)$ in $-\pi \leq x < 0$ by the equation $f(-x)=-f(x)$. The function thus defined for $(-\pi, \pi)$

satisfies Dirichlet's conditions in this interval. And it may be easily seen that

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \text{ leads to } b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt,$$

and that $a_n = 0$ ($n = 0, 1, 2, \dots$).

Hence the Fourier's series for $f(x)$ is now

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \int_0^{\pi} f(t) \sin nt \, dt.$$

Thus when $f(x)$ is an arbitrary function satisfying Dirichlet's conditions in $(0, \pi)$, the sum of the half-range sine series equal to $\frac{1}{2} [f(x+0) + f(x-0)]$ at every point between 0 and π where $f(x+0)$ and $f(x-0)$ exist; and since all the terms of the series vanish when $x=0$ and $x=\pi$, the sum is zero at these points.

Note 1. It will be noticed that when $f(x)$ is continuous at the end points $x=0$ and $x=\pi$, the cosine series gives the value of the function at these points. The sine series gives the value of $f(x)$ at these points if $f(x)$ is zero there.

Note 2. The sum functions of the half-range sine and cosine series are periodic with period 2π .

Note 3. Consider the three cases :

- (i) $f(x)$ is defined in the interval $(-\pi, \pi)$,
- (ii) $f(x)$ is defined in $(0, \pi)$ and extended to $(-\pi, 0)$ by the equation $f(-x) = f(x)$,
- (iii) $f(x)$ is defined in $(0, \pi)$ and extended to $(-\pi, 0)$ by the equation $f(-x) = -f(x)$;

whereas the Fourier's series for $f(x)$ in the above three cases have identical sums for values of x in the open interval $(0, \pi)$, their sums are different for values of x in the open interval $(-\pi, 0)$.

15.7. Transformations.

- (a) To change the interval $(-\pi, \pi)$ to $(-l, l)$.

Take the Fourier's series for $\phi(y)$ in $(-\pi, \pi)$ in the form

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(v) dv + \frac{1}{\pi} \sum_1^{\infty} \left\{ \cos ny \int_{-\pi}^{\pi} \phi(x) \cos nv dv \right. \\ \left. + \sin ny \int_{-\pi}^{\pi} \phi(v) \sin nv dv \right\};$$

and put $y = \frac{\pi x}{l}$ and $v = \frac{\pi t}{l}$; the limits are thereby changed into $-l, l$.

Now $\phi(y) = \phi\left(\frac{\pi x}{l}\right) = f(x)$ say and so $\phi(v) = \phi\left(\frac{\pi t}{l}\right) = f(t)$. Also

$$\frac{1}{\pi} \cos ny \int_{-\pi}^{\pi} \phi(v) \cos nv dv \text{ is transformed into} \\ \frac{1}{l} \cos \frac{n\pi x}{l} \int_{-l}^l \phi\left(\frac{\pi t}{l}\right) \cos \frac{n\pi t}{l} dt = \frac{1}{l} \cos \frac{n\pi x}{l} \int_{-l}^l f(t) \cos \frac{n\pi t}{l} dt,$$

and similarly $\frac{1}{\pi} \sin ny \int_{-\pi}^{\pi} \phi(v) \sin nv dv$ is transformed into

$$\frac{1}{l} \sin \frac{n\pi x}{l} \int_{-l}^l f(t) \sin \frac{n\pi t}{l} dt.$$

Thus the series is transformed into

$$\frac{1}{2l} \int_{-l}^l f(t) dt + \frac{1}{l} \sum_1^{\infty} \left\{ \cos \frac{n\pi x}{l} \int_{-l}^l f(t) \cos \frac{n\pi t}{l} dt \right. \\ \left. + \sin \frac{n\pi x}{l} \int_{-l}^l f(t) \sin \frac{n\pi t}{l} dt \right\}, (-l \leq x \leq l).$$

The corresponding cosine and sine series are

$$\frac{1}{l} \int_0^l f(t) dt + \frac{2}{l} \sum_1^{\infty} \cos \frac{n\pi x}{l} \int_0^l f(t) \cos \frac{n\pi t}{l} dt, (0 \leq x \leq l)$$

$$\frac{2}{l} \sum_1^{\infty} \sin \frac{n\pi x}{l} \int_0^l f(t) \sin \frac{n\pi t}{l} dt, (0 \leq x \leq l).$$

It is to be understood that in all the three cases, $f(x)$ must satisfy Dirichlet's conditions in $(-l, l)$.

(b) To change the interval $(-\pi, \pi)$ into $(0, 2\pi)$.

Taking the Fourier's series for $\phi(y)$ in $(-\pi, \pi)$ as

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(v) dv + \frac{1}{\pi} \sum_1^{\infty} \left\{ \cos ny \int_{-\pi}^{\pi} \phi(v) \cos nv dv \right. \\ \left. + \sin ny \int_{-\pi}^{\pi} \phi(v) \sin nv dv \right\}, \end{aligned}$$

we put $y = x - \pi$, $v = t - \pi$; the limits are at once changed into 0, 2π .

Now $\phi(y) = \phi(x - \pi) = f(x)$ say and so $\phi(v) = \phi(t - \pi) = f(t)$.

And $\frac{1}{\pi} \cos ny \int_{-\pi}^{\pi} \phi(v) \cos nv dv$ is changed into

$$\begin{aligned} \frac{1}{\pi} \cos n(x - \pi) \int_0^{2\pi} \phi(t - \pi) \cos n(t - \pi) dt \\ = (-1)^n \frac{1}{\pi} \cos nx \int_0^{2\pi} (-1)^n f(t) \cos nt dt \\ = \frac{1}{\pi} \cos nx \int_0^{2\pi} f(t) \cos nt dt. \end{aligned}$$

Similarly $\frac{1}{\pi} \sin ny \int_{-\pi}^{\pi} \phi(v) \sin nv dv$ is changed into

$$= \frac{1}{\pi} \sin nx \int_0^{2\pi} f(t) \sin nt dt.$$

Hence the Fourier's series for $f(x)$ in the interval $(0, 2\pi)$ takes the form

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) dt + \frac{1}{\pi} \sum_1^{\infty} \left\{ \cos nx \int_0^{2\pi} f(t) \cos nt dt \right. \\ \left. + \sin nx \int_0^{2\pi} f(t) \sin nt dt \right\}, \quad (0 \leq x \leq 2\pi),$$

provided $f(x)$ satisfies Dirichlet's conditions in $(0, 2\pi)$.

(c) To change the interval $(-\pi, \pi)$ into (a, b) , ($a < b$).

Starting with the Fourier's series for $\phi(y)$ in $(-\pi, \pi)$ in the form,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(v) dv + \frac{1}{\pi} \sum_1^{\infty} \left\{ \cos ny \int_{-\pi}^{\pi} \phi(v) \cos nv dv \right. \\ \left. + \sin ny \int_{-\pi}^{\pi} \phi(v) \sin nv dv \right\},$$

we put $y = \frac{2\pi(x-a)}{b-a} - \pi$ and $v = \frac{2\pi(t-a)}{b-a} - \pi$; then the limits $-\pi, \pi$ are at once changed into a and b .

Now $\phi(y) = \phi \left\{ \frac{2\pi(x-a)}{b-a} - \pi \right\} = f(x)$ say, and so $\phi(v) = f(t)$.

$$\text{Also } \frac{1}{\pi} \cos ny \int_{-\pi}^{\pi} \phi(v) \cos nv dv$$

$$= \frac{2}{b-a} \cos \left\{ \frac{2n\pi(x-a)}{b-a} - n\pi \right\} \int_a^b f(t) \cos \left\{ \frac{2n\pi(t-a)}{b-a} - n\pi \right\} dt$$

$$= \frac{2}{b-a} \cos \frac{2n\pi(x-a)}{b-a} \int_a^b f(t) \cos \frac{2n\pi(t-a)}{b-a} dt;$$

$$\text{and similarly } \frac{1}{\pi} \sin ny \int_{-\pi}^{\pi} \phi(v) \sin nv dv$$

$$= \frac{2}{b-a} \sin \frac{2n\pi(x-a)}{b-a} \int_a^b f(t) \sin \frac{2n\pi(t-a)}{b-a} dt.$$

$$\begin{aligned} \text{Again } \cos \frac{2n\pi(x-a)}{b-a} \int_a^b f(t) \cos \frac{2n\pi(t-a)}{b-a} dt \\ + \sin \frac{2n\pi(x-a)}{b-a} \int_a^b f(t) \sin \frac{2n\pi(t-a)}{b-a} dt \\ = \int_a^b f(t) \cos \frac{2n\pi(t-x)}{b-a} dt. \end{aligned}$$

Thus the Fourier's series for $\phi(y)$ in $(-\pi, \pi)$ is transformed into

$$\frac{1}{b-a} \int_a^b f(t) dt + \frac{2}{b-a} \sum_{n=1}^{\infty} \int_a^b f(t) \cos \frac{2n\pi(t-x)}{b-a} dt,$$

$$(a \leq x \leq b)$$

on the understanding that $f(x)$ satisfies Dirichlet's conditions in (a, b) .

15.8. Calculation of Fourier's coefficients

The following examples will show the method of calculation of Fourier's coefficients for a given function.

1. Find a series of sines and cosines of multiples of x which will represent $x+x^2$ in the interval $-\pi < x < \pi$,

(C. H. 1960, '62, '64)

$$\text{Here } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^3}{3},$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx,$$

and after integration by parts, we find that $a_n = \frac{4}{n^3} \cos n\pi$.

$$\text{Also } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$\text{which gives } b_n = (-1)^{n-1} \frac{2}{n}.$$

Putting these values in

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots,$$

$$\text{we get } x+x^2 = \frac{\pi^2}{3} + 4 \left(-\cos x + \frac{1}{2} \sin x \right) - 4 \left(\frac{-1}{2^2} \cos 2x + \frac{1}{4} \sin 2x \right) \\ + \dots \text{ when } -\pi < x < \pi.$$

$$\text{At } x = \pm \pi, \frac{1}{2} [f(-\pi+0) + f(\pi-0)] \text{ gives for}$$

$$f(x) = x+x^2 \text{ the value } \pi^2.$$

$$\therefore \pi^2 = \frac{\pi^2}{3} + 4 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right)$$

$$\text{or } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots.$$

Note. Since x , $x \cos nx$, $x^2 \sin nx$ are odd functions, we can omit these in calculating a_0 , a_n , b_n for

$$\int_{-\pi}^{\pi} x \, dx, \int_{-\pi}^{\pi} x \cos nx \, dx, \int_{-\pi}^{\pi} x^2 \sin nx \, dx \text{ all van'ish.}$$

2. Find a series of sines and cosines of multiples of x which will represent $f(x)$ in the interval $-\pi < x < \pi$, when

$$f(x) = 0, (-\pi < x \leq 0)$$

$$f(x) = \frac{\pi x}{4}, (0 < x < \pi).$$

(C. H. 1963, 1964)

We first observe that

$$\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^0 f(x) \, dx + \int_0^{\pi} f(x) \, dx = \int_0^{\pi} f(x) \, dx,$$

since $f(x) = 0$ in $(-\pi, 0)$.

Hence $a_0 = \frac{1}{\pi} \int_0^{\pi} \frac{\pi x}{4} dx = \frac{\pi^2}{8},$

$$a_n = \frac{1}{\pi} \int_0^{\pi} \frac{\pi x}{4} \cos nx \, dx = \frac{1}{4n^2} (\cos n\pi - 1),$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} \frac{\pi x}{4} \sin nx \, dx = -\frac{\pi}{4n} \cos nx.$$

$$\therefore f(x) = \frac{\pi^2}{16} + \frac{1}{2} \left(-\cos x + \frac{\pi}{2} \sin x \right) - \frac{\pi}{8} \sin 2x + \dots$$

when $-\pi < x < \pi.$

At $x = \pm\pi$, $\frac{1}{2} \left[f(-\pi+0) + f(\pi-0) \right] = \frac{\pi^2}{8}$ and so we get

$$\frac{\pi^2}{8} = \frac{\pi^2}{16} + \frac{1}{2} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right),$$

$$\text{or, } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

3. Expand x^2 in Fourier's series in $-\pi < x < \pi.$

Obviously x^2 is an even function of x . So the expansion will be in a cosine series, namely

$$\frac{1}{\pi} \int_0^{\pi} t^2 \, dt + \frac{2}{\pi} \sum_1^{\infty} \cos nx \int_0^{\pi} t^2 \cos nt \, dt.$$

$$\text{Now } \frac{1}{\pi} \int_0^{\pi} t^2 \, dt = \frac{\pi^2}{3}.$$

Also by integration by parts, $\int_0^{\pi} t^2 \cos nt \, dt = (-1)^n \frac{2\pi}{n^2}.$

$$\text{Hence } x^2 = \frac{\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right), \quad (-\pi < x < \pi.)$$

At $x = \pm \pi$, the sum of the series is π^2 and we get the result

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

4. Expand x in a Fourier's series in $-\pi < x < \pi$. (C. H. 1962)
 x being an odd function of x , the expansion will be in a sine series. It is

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \int_0^{\pi} t \sin nt \, dt.$$

By integration by parts,

$$\int_0^{\pi} t \sin nt \, dt = (-1)^{n+1} \frac{\pi}{n}.$$

$$\therefore x = 2 \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right) \quad \text{when } -\pi < x < \pi.$$

At $x = \pm \pi$, the sum of the series is obviously 0.

Putting $x = \frac{\pi}{2}$, we get $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

5. If $f(x) = x$ in $(0, \pi)$, expand $f(x)$ in a cosine series.

Defining $f(x)$ in $(-\pi, 0)$ by the equation $f(-x) = f(x)$, the required series is

$$\frac{1}{\pi} \int_0^{\pi} t \, dt + \frac{2}{\pi} \sum_{n=1}^{\infty} \cos nx \int_0^{\pi} t \cos nt \, dt.$$

Now $\int_0^{\pi} t \cos nt \, dt = \frac{1}{n^2} [(-1)^n - 1]$; hence

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right].$$

Note. If $f(x) = |x|$ in $-\pi < x < \pi$, we also get the same expansion.
 $x=0$ gives,

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Examples.

1. Find the Fourier's series for the function defined by

$$\begin{aligned} f(x) &= 1 \quad \text{for } 0 < x < \pi, \\ &= 0 \quad \text{for } x = 0, \\ &= -1 \quad \text{for } -\pi < x < 0. \end{aligned}$$

Deduce the value of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

[The given function is an odd function and so expands into the sine series :

$$\begin{aligned} & \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \int_0^{\pi} \sin nt \, dt \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx. \end{aligned}$$

$$\text{So } f(x) = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right).$$

$$\text{Putting } x = \frac{\pi}{2}, \text{ we get } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots]$$

2. Expand in a series of sines and cosines of multiples of
- x
- , the function defined by

$$\begin{aligned} f(x) &= \pi - x \quad \text{when } 0 < x < \pi, \\ &= x - \pi \quad \text{when } -\pi < x < 0. \end{aligned}$$

What is the value of the series for $x = \pm\pi$ and $x = 0$?

(C. H. 1966)

$$\begin{aligned} \left[a_n &= \frac{1}{\pi} \int_{-\pi}^0 f(t) \cos nt \, dt + \frac{1}{\pi} \int_0^{\pi} f(t) \cos nt \, dt \right. \\ &= \frac{1}{\pi} \int_{-\pi}^0 (t - \pi) \cos nt \, dt + \frac{1}{\pi} \int_0^{\pi} (x - t) \cos nt \, dt \\ &= \frac{2\{1 - (-1)^n\}}{n^2 \pi} \quad (n \neq 0); \end{aligned}$$

$$\text{and } a_0 = -\pi.$$

$$\text{Similarly } b_n = \frac{1}{\pi} \int_{-\pi}^0 (t-\pi) \sin nt \, dt + \frac{1}{\pi} \int_0^{\pi} (\pi-t) \sin nt \, dt$$

$$= \frac{2\{1 - (-1)^n\}}{n}.$$

a_n and b_n both vanish when n is even. Thus

$$f(x) = -\frac{1}{2}\pi + \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\pi \sin x}{1} + \frac{\cos 3x}{3^2} + \frac{\pi \sin 3x}{3} + \dots \right)$$

$$= -\frac{1}{2}\pi + \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots \right) + 4 \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \dots \right).$$

At $x=0$, the sum is $\frac{1}{2} [f(+0) + f(-0)] = \frac{1}{2} [(x) + (-\pi)] = 0$.

At $x = \pm \pi$, the sum

$$= \frac{1}{2} [f(\pi-0) + f(-\pi+0)] = \frac{1}{2} [0 + (-2\pi)] = -\pi.$$

Putting any of these values, we get

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots.]$$

3. Find the sine series which represents $f(x) = \pi - x$ in $0 < x < \pi$.
(C. H. 1966)

$$\left[b_n = \frac{2}{\pi} \int_0^{\pi} (\pi-t) \sin nt \, dt = \frac{2}{n} \right]$$

Since $f(x)$ is continuous in $0 < x < \pi$,

$$\pi - x = 2 \left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right).$$

The right side is zero for both $x=0$ and $x=\pi$ and the left side for only $x=\pi$. The representation holds for $x=\pi$ but not for $x=0$.]

4. Find the Fourier's cosine series for $f(x) = \pi - x$ in $0 < x < \pi$. Prove that the sum of the series for $x = \pm \pi$ is 0. Hence deduce the sum of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

(C. H. 1961)

[$f(x)$ being continuous in $0 < x < \pi$, we get

$$\begin{aligned}\pi - x &= \frac{\pi}{2} + \sum \frac{2\{1 - (-1)^n\}}{\pi n^2} \cos nx \\ &= \frac{\pi}{2} + \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots \right).\end{aligned}$$

The sum of the series at $x = 0$ is $f(+0) = \pi$ and at $x = \pi$, it is $f(\pi - 0) = 0$ whence we get $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$]

5. Show that if $0 < x < \pi$, then

$$\pi - x = \frac{\pi}{2} + \sum_1^{\infty} \frac{\sin 2nx}{n}.$$

Show that the equation does not hold for $x = 0$ and $x = \pi$ and explain why it does not hold. (C. H. 1963)

[When the limits are changed to a, b , the Fourier's series for $f(x)$ in (a, b) is

$$\begin{aligned}\frac{1}{b-a} \int_a^b f(t) dt + \frac{2}{b-a} \sum \cos \frac{2n\pi(x-a)}{b-a} \int_a^b f(t) \cos \frac{2n\pi(t-a)}{b-a} dt \\ + \frac{2}{b-a} \sum \sin \frac{2n\pi(x-a)}{b-a} \int_a^b f(t) \sin \frac{2n\pi(t-a)}{b-a} dt.\end{aligned}$$

[Art. 15.7 (c)]

In this make $a = 0$ and $b = \pi$. We easily get

$$\int_a^b f(t) \cos \frac{2n\pi(t-a)}{b-a} dt = 0, \quad \int_a^b f(t) \sin \frac{2n\pi(t-a)}{b-a} dt = \frac{\pi}{2n},$$

$$\text{and } \int_a^b f(t) dt = \frac{\pi^2}{2}.$$

$$\text{Thus } \pi - x = \frac{\pi}{2} + \frac{\sin 2x}{1} + \frac{\sin 4x}{2} + \frac{\sin 6x}{3} + \dots]$$

6. Given $f(x) = \frac{1}{4} \pi x$ when $0 \leq x \leq \frac{\pi}{2}$,

$f(x) = \frac{1}{4} \pi(\pi - x)$ when $\frac{\pi}{2} \leq x \leq \pi$,

find (i) a series of cosines of multiples of x which will represent $f(x)$ in the interval $0 \leq x \leq \pi$;

(ii) a series of sines of multiples of x which will represent $f(x)$ in the interval $0 \leq x \leq \pi$. (C. H. 1965)

$$\begin{aligned} \left[(i) \quad a_n &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{4} \pi x \cos nx \, dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{4} \pi(\pi - x) \cos nx \, dx \right. \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} x \cos nx \, dx + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \cos nx \, dx, \end{aligned}$$

which gives $a_n = -\frac{1}{2n^2} \left\{ 1 + \cos n\pi - 2 \cos \frac{n\pi}{2} \right\} = \frac{2}{n^2} \cos \frac{n\pi}{2} \sin^2 \frac{n\pi}{4}$.

Also $\frac{2}{n^2} \cos \frac{n\pi}{2} \sin^2 \frac{n\pi}{4} = 2 \cos \frac{n\pi}{2} \cdot \left(\frac{\sin \frac{n\pi}{4}}{\frac{n\pi}{4}} \right)^2 \cdot \frac{\pi^2}{16}$;

as $n \rightarrow 0$, we get $a_0 = \frac{\pi^2}{8}$.

$\therefore f(x) = \frac{\pi^2}{16} - 2 \left\{ \frac{\cos 2x}{2^2} + \frac{\cos 6x}{6^2} + \dots \right\}, (0 \leq x \leq \pi)$

$$\begin{aligned} (ii) \quad b_n &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{4} \pi x \sin nx \, dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{4} \pi(\pi - x) \sin nx \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} x \sin nx \, dx + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx \, dx \end{aligned}$$

giving $b_n = \frac{1}{n^2} \sin \frac{n\pi}{2}$. Thus

$f(x) = \sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots$

7. Find the Fourier series for $f(x)$ where

$$f(x) = \frac{\pi}{2 \sinh \pi} \cdot e^x$$

in the interval $-\pi < x < \pi$. What is the sum of the series for $x = \pm \pi$? (C. H. 1965)

$$\left[a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\pi}{2 \sinh \pi} \cdot e^t \cos nt \, dt \right.$$

$$= \frac{1}{2 \sinh \pi} \int_{-\pi}^{\pi} e^t \cos nt \, dt = \frac{(-1)^n}{1+n^2};$$

$$b_n = \frac{1}{2 \sinh \pi} \int_{-\pi}^{\pi} e^t \sin nt \, dt = (-1)^{n-1} \frac{n}{1+n^2}; \text{ also } a_0 = 1.$$

$$\therefore \frac{\pi}{2 \sinh \pi} e^x = \frac{1}{2} + \left(-\frac{1}{1+1^2} \cos x + \frac{1}{1+1^2} \sin x \right) \\ + \left(\frac{1}{1+2^2} \cos 2x - \frac{2}{1+2^2} \sin 2x \right) + \dots \quad (-\pi < x < \pi).$$

$$\text{At } x = \pm \pi, \frac{1}{2} [f(-\pi+0) + f(\pi-0)] = \frac{\pi}{2 \sinh \pi} \cdot (e^{\pi} + e^{-\pi}) \\ = \frac{\pi}{2} \coth \pi.]$$

8. Represent $f(x)$ where $f(x) = \cos px$ in $-\pi \leq x \leq \pi$ (p not being an integer) in a Fourier Series. Deduce that

$$\frac{\pi}{\sin p\pi} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+p} + \frac{1}{n+1-p} \right). \quad (\text{C. H. 1965})$$

[Since $\cos px$ is an even function, the expansion will be in a cosine series.

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos pt \cos nt \, dt = \frac{1}{\pi} \int_0^{\pi} \{ \cos (p+n)t + \cos (p-n)t \} dt \\ = \frac{\sin (p+n)\pi}{(p+n)\pi} + \frac{\sin (p-n)\pi}{(p-n)\pi};$$

whence $a_0 = 2 \cdot \frac{\sin p\pi}{p\pi}$. Now it is easy to see that

$$\sin(p+n)\pi = \sin(p-n)\pi = (-1)^n \sin p\pi.$$

$$\therefore \frac{\sin(p+n)\pi}{(p+n)\pi} + \frac{\sin(p-n)\pi}{(p-n)\pi} = \frac{(-1)^n 2p \sin p\pi}{\pi(p^2 - n^2)}.$$

$$\text{So } \cos px = \frac{\sin p\pi}{\pi} \left\{ \frac{1}{p} - \frac{2p \cos x}{p^2 - 1^2} + \frac{2p \cos 2x}{p^2 - 2^2} - \frac{2p \cos 3x}{p^2 - 3^2} + \dots \right\} \quad (-\pi \leq x \leq \pi).$$

Putting $x=0$, we get

$$1 = \frac{\sin p\pi}{\pi} \left\{ \frac{1}{p} - \frac{1}{p+1} - \frac{1}{p-1} + \frac{1}{p+2} + \frac{1}{p-2} - \frac{1}{p+3} - \frac{1}{p-3} + \dots \right\}$$

$$\therefore \frac{\pi}{\sin p\pi} = \sum_0^\infty (-1)^n \left(\frac{1}{n+p} + \frac{1}{n+1-p} \right).$$

Note. Putting $x=\pi$, we deduce $\pi \cot p\pi = \frac{1}{p} + 2p \sum \frac{1}{p^2 - n^2}$.

9. When x lies between $\pm \pi$, prove the relations :

$$\sin mx = \frac{2}{\pi} \sin mx \left(\frac{\sin x}{1^2 - m^2} - \frac{2 \sin 2x}{2^2 - m^2} + \frac{3 \sin 3x}{3^2 - m^2} - \dots \right),$$

$$\cos mx = \frac{2}{\pi} \sin mx \left(\frac{1}{2m} + \frac{m \cos x}{1^2 - m^2} - \frac{m \cos 2x}{2^2 - m^2} + \frac{m \cos 3x}{3^2 - m^2} - \dots \right).$$

Hence prove the relation

$$\cot u = \frac{1}{u} + \frac{2u}{u^2 - \pi^2} + \frac{2u}{u^2 - 4\pi^2} + \dots$$

[To prove the last relation, put $x=\pi$ and $m\pi=u$.]

10. For all values of x between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, prove that

$$x = \frac{4}{\pi} \left(\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \right)$$

[From Ex. 5, Art. 15'8, we have, when x lies between 0 and π ,

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right]$$

Now put $x = \frac{\pi}{2} - y$.]

11. Find a function of x which has the value c when x lies between 0 and a , and the value 0 when x lies between a and l .

[Use the cosine series of Art. 15'7(a),

$$\phi(x) = \frac{ca}{l} + \frac{2c}{\pi} \left[\sin \frac{\pi a}{l} \cos \frac{\pi x}{l} + \frac{1}{2} \sin \frac{2\pi a}{l} \cos \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi a}{l} \cos \frac{3\pi x}{l} + \dots \right]$$

12. Find a function which shall be equal to $\cos x$ for all values of x between 0 and π , and to $-\cos x$ for all values of x between $-\pi$ and 0.

[We easily find $\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0$, and we get

$$\cos x = \frac{4}{\pi} \left\{ \frac{2}{1.3} \sin 2x + \frac{4}{3.5} \sin 4x + \frac{6}{5.7} \sin 6x + \dots \right\}.]$$

13. Prove that when $-1 < x < 1$,

$$x + x^2 = \frac{1}{3} + \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{2 \cos n\pi x}{n^2 \pi} - \frac{\sin n\pi x}{n} \right\}.$$

[Take the series in Art. 15'7 (c) and put $a = -1$, $b = 1$.]

14. Find the Fourier series which represents $|\sin x|$ in $-\pi \leq x \leq \pi$.

$$\left[\frac{2}{\pi} - \sum \frac{4 \cos 2nx}{(4n^2 - 1)\pi} \right]$$

15. If $f(x)$ equals x in $\left(0, \frac{\pi}{2}\right)$, and equals 0 in $\left(\frac{\pi}{2}, \pi\right)$, then prove that

$$f(x) = \frac{2}{\pi} \left\{ \frac{\sin x}{1^2} + \frac{\pi}{2^2} \sin 2x - \frac{\sin 3x}{3^2} - \frac{2\pi}{4^2} \sin 4x + \frac{\sin 5x}{5^2} + \frac{3\pi}{6^2} \sin 6x - \dots \right\}.$$

16. Prove that the equation, in rectangular coordinates,

$$y = \frac{2}{3}h + \frac{4h}{\pi^2} \left(\cos \frac{\pi x}{l} - \frac{1}{2^2} \cos \frac{2\pi x}{l} + \frac{1}{3^2} \cos \frac{3\pi x}{l} - \dots \right)$$

represents a series of equal and similar parabolic arcs of height h and span $2k$ standing in contact along the axis of x .

[If P be the point (x, y) , we have $x^2 = \lambda(h - y)$ as the equation of the parabola.

Also $y = 0$ when $x = k$;

$$\therefore k^2 = \lambda h.$$

$$\text{Hence } h - y = \frac{x^2}{\lambda} = \frac{h}{k^2} \cdot x^2.$$

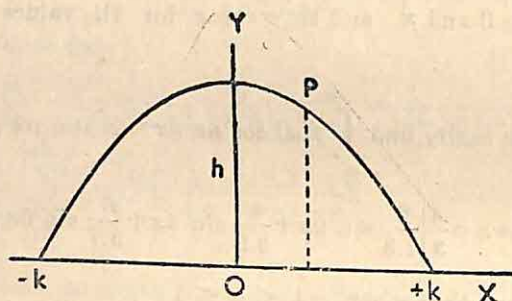


Fig. 17

The Fourier series for this is, since it is an even function,

$$h - y = \frac{1}{k} \int_0^k \frac{h}{k^2} t^2 dt + \frac{2}{k} \sum \cos \frac{n\pi x}{k} \int_0^k \frac{h}{k^2} t^2 \cos \frac{n\pi t}{k} dt.$$

Evaluating the integrals, we get

$$h - y = \frac{h}{3} + \frac{4h}{\pi^2} \left\{ -\cos \frac{\pi x}{k} + \frac{1}{2^2} \cos \frac{2\pi x}{k} - \frac{1}{3^2} \cos \frac{3\pi x}{k} + \dots \right\}$$

$$\text{or, } y = \frac{2h}{3} + \frac{4h}{\pi^2} \left\{ \cos \frac{\pi x}{k} + \frac{1}{2^2} \cos \frac{2\pi x}{k} + \frac{1}{3^2} \cos \frac{3\pi x}{k} - \dots \right\}.$$

CHAPTER 16

FUNCTIONS OF SEVERAL VARIABLES

(LIMIT & CONTINUITY)

16'1. A function of two variables.

If corresponding to definite values of the variables x and y such that $a \leq x \leq b$ and $c \leq y \leq d$, the variable z has got a definite value, z is called a function of the two variables x and y , defined in the domain $(a, b; c, d)$. x and y are the independent variables and z is the dependent variable and the relation between the variables x, y, z is expressed in the form $z=f(x, y)$.

As in the case of functions of one variable only, for functions of more than one independent variable, we say that a function is *single-valued* or *many-valued* according as the dependent variable takes one or more values for a given set of values of the independent variables. And as before, we define polynomials, rational functions, explicit and implicit algebraical functions, transcendental functions, etc., in the same way. Again just as the explicit function $y=f(x)$ of a single variable x may be looked upon as a particular case of the more general implicit relation $f(x, y)=0$, so the explicit function $z=f(x, y)$ of the two variables x, y is a special case of the more general implicit relation $f(x, y, z)=0$, x, y being independent variables and z dependent upon them.

The domain $(a, b; c, d)$ in which x and y vary is called the *domain of existence* of the function and the set of values of z is called the *range* of the function.

Note. The set of all ordered number-pairs (x, y) of real numbers is called a *space of two demensions* and denoted by the symbol E_2 . The aggregate of ordered number-pairs which form the domain of $f(x, y)$ is a sub-set S of E_2 . We can now say that $f(x, y)$ is defined in the sub-set S of the set E_2 .

16'2. A function of several variables.

A function u of several independent variables $x, y, z, \dots$ is defined in an exactly similar manner and is written

$$u=f(x, y, z, \dots) \text{ or } \phi(u, x, y, z, \dots)=0.$$

16'3. Geometrical representation of a function of two variables.

We can represent geometrically a function z of two variables, x and y , defined in the domain $(a, b; c, d)$ by taking three axes OX, OY, OZ in three dimensional space and defining a point (x, y, z) as a point in space of which the coordinates are x, y, z . As x and y take different values in the plane XOY , z takes positions in space, the aggregate of such positions being the locus $z=f(x, y)$. A set of values of x and y gives a point in the plane XOY and if $a \leq x \leq b$ and $c \leq y \leq d$, this point lies on or within the parallelogram bounded by the lines $x=a, x=b; y=c, y=d$; for every point in this domain, z has got a definite value. The domain, however, need not be a rectilineal figure. Those acquainted with the elements of solid geometry will easily see that the locus of $z=f(x, y)$ is a surface.

16'4. Simultaneous or Double Limit.

The function $f(x, y)$ is said to tend the limit L as the point (x, y) tends to the point (a, b) , that is to say, as x tends to a and y to b simultaneously and in any manner whatsoever, if corresponding to any arbitrary positive number ϵ , however small, a positive number δ (depending on ϵ) can be found such that

$$|f(x, y) - L| < \epsilon$$

for values of x and y satisfying $|x-a| \leq \delta, |y-b| \leq \delta$.

The limit L , if it exists, is called the *simultaneous* or *double* limit of $f(x, y)$ as (x, y) tends to (a, b) .

Thus when the limit exists, for all points on or within the square of centre (a, b) and whose sides are of length 2δ , [except possibly the point (a, b)], the values of $f(x, y)$ will differ from L by less than ϵ . Such a square is called a *neighbourhood* of the point (a, b) . Clearly we could substitute for the square, a circle of centre (a, b) and radius ρ and in this case we would have to replace the statements $|x-a| \leq \delta, |y-b| \leq \delta$ by $\sqrt{(x-a)^2 + (y-b)^2} \leq \rho$. And we write

$$\text{Lt}_{(x, y) \rightarrow (a, b)} f(x, y) = L,$$

or

$$\begin{array}{l} \text{Lt} \\ x \rightarrow a \\ y \rightarrow b \end{array} f(x, y) = L.$$

16'5. Repeated or Iterated Limits.

Let $f(x, y)$ be defined in a certain neighbourhood of (a, b) . Then

$$\lim_{x \rightarrow a} f(x, y)$$

if it exists, is a function of y , say $\phi(y)$. If now

$$\lim_{y \rightarrow b} \phi(y)$$

exists and is equal to l , we write

$$\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y) = l.$$

l is then called a *repeated* or *iterated* limit of $f(x, y)$ as $x \rightarrow a$ and $y \rightarrow b$. On changing the order of passing to limits, i.e., if first $y \rightarrow b$ and then $x \rightarrow a$, we may have a different result, provided the limit exists. In other words,

$$\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y) \text{ may not be equal to } \lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y).$$

For example

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x-y}{x+y} = \lim_{y \rightarrow 0} \frac{-y}{y} = -1,$$

$$\text{but } \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x-y}{x+y} = \lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

Observation : It can be easily proved that if the double limit exists, the repeated limits exist and the three limits have the same value. If however the repeated limits exist and are unequal, the double limit cannot exist. Further even if the repeated limits exist and are equal, the double limit may or may not exist.

16'6. Necessary and sufficient condition for the existence of a limit.

It is easy to deduce from the definition of a limit that a necessary condition for the existence of the limit

$$(x, y) \rightarrow (a, b) \quad f(x, y)$$

is that corresponding to an arbitrarily chosen positive number ϵ , however small, there is a positive number δ such that

$$|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$$

when (x_1, y_1) and (x_2, y_2) are any two points in the δ -neighbourhood of (a, b) with the possible exception of the point (a, b) .

The condition is also sufficient for the existence of a limit.

16.7. Theorems on Limits.

(a) Theorem I.

If $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = A$ and $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y) = B$, then

$$\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) + \phi(x, y)] = A + B.$$

As $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = A$, corresponding to a pre-assigned positive

ϵ , there is a positive number δ_1 such that

$$|f(x, y) - A| < \frac{\epsilon}{2} \text{ when } |x - a| \leq \delta_1, |y - b| \leq \delta_1, \dots (1)$$

Similarly as $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y) = B$, corresponding to the same ϵ

there is a positive number δ_2 such that

$$|\phi(x, y) - B| < \frac{\epsilon}{2} \text{ when } |x - a| \leq \delta_2, |y - b| \leq \delta_2, \dots (2)$$

If δ be smaller of δ_1 and δ_2 , both (1) and (2) hold when $|x - a| \leq \delta, |y - b| \leq \delta$. Now,

$$\begin{aligned} |f(x, y) + \phi(x, y) - (A + B)| &= |f(x, y) - A + \phi(x, y) - B| \\ &\leq |f(x, y) - A| + |\phi(x, y) - B| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) + \phi(x, y)] = A + B$.

Cor. 1. It can be shown similarly that

$$\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) - \phi(x, y)] = A - B.$$

Cor. 2. The theorem can be proved to hold for any finite number of functions.

(b) Theorem II.

If $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = A$ and $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} \phi(x, y) = B$, then

$$\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} [f(x, y) \cdot \phi(x, y)] = A \cdot B.$$

Using the relation $\phi - AB = (f - A)(\phi - B) + A(\phi - B) + B(f - A)$ and following the method adopted in the case of functions of one variable, [Art. 58 (ii)] the theorem is easily proved.

(c) Theorem III.

Under the same hypotheses as in (a) above,

$$\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow b}} \left[\frac{f(x, y)}{\phi(x, y)} \right] = \frac{A}{B}, \text{ proved } B \neq 0.$$

By following closely Art. 58 (iii), the theorem is easily established.

Cor. $\text{Lt}_{\substack{x \rightarrow a \\ y \rightarrow a}} \left[\frac{1}{\phi(x, y)} \right] = \frac{1}{B}, \text{ unless } B = 0.$

16'8. Continuity.**Continuity at a point.**

A function $f(x, y)$ is said to be continuous at the point (a, b) if (i) $f(a, b)$ exists, (ii) $f(x, y)$ exists at all points of a certain neighbourhood of (a, b) , and (iii) $f(x, y)$ tends to $f(a, b)$ as the point (x, y) tends to the point (a, b) in any manner.

Alternatively, $f(x, y)$ is continuous at (a, b) , if corresponding to any preassigned positive number ϵ however small, there is a positive number δ depending on ϵ such that

$$|f(x, y) - f(a, b)| < \epsilon$$

whenever $|x - a| \leq \delta, |y - b| \leq \delta.$

It should be observed that to assert the continuity of a function of two variables is to assert more than its continuity with respect to each variable separately. If $f(x, y)$ is a continuous function of two variables, it will be continuous with respect to each variable separately when any fixed value is given to the other. But

the converse is not necessarily true. The following example will make this clear.

Consider the function

$$f(x, y) = \frac{2xy}{x^2 + y^2} \text{ when } x \neq 0, y \neq 0, \\ = 0 \text{ when either } x \text{ or } y \text{ or both } = 0.$$

Putting $y = q$, $f(x, q) = \frac{2xq}{x^2 + q^2}$ is continuous for all values of x including $x = 0$.

Again putting $x = p$, $f(p, y) = \frac{2py}{p^2 + y^2}$ is continuous for all values of y including $y = 0$.

If x and y tend to 0 along the line $y = mx$, $f(x, y) = \frac{2m}{1+m^2}$ and

$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = \frac{2m}{1+m^2}$ which may have any value between -1

and 1 and is thus not necessarily equal to $f(0, 0)$ which is zero by definition, proving discontinuity at $(0, 0)$.

However it is continuous in any domain which does not include the origin.

Continuity in a given domain.

A function $f(x, y)$ is continuous in a given domain if it is continuous at every point of the domain.

Continuity of composite functions.

Applying the limit theorems (Art. 16'7) it is easy to prove that (i) the sum, difference and product of two functions each continuous at a given point or in a given domain, is continuous at the point or in the domain;

(ii) the quotient of two functions, each continuous at a given point or in a given domain, is continuous at the point or in the domain, except at points where the denominator vanishes;

(iii) a continuous function of a continuous function is itself a continuous function,

16'9. Properties of a continuous function of two variables.

We state below without proof some properties of a continuous function of two variables :

(a) If the function $f(x, y)$ is not bounded in a closed domain D , we can find a point (ξ, η) in the domain where $f(x, y)$ is not bounded. As a consequence $f(x, y)$ is discontinuous at this point of domain.

(b) A function continuous in a closed domain is bounded therein and attains its bounds at least once in the domain.

(c) If $f(x, y)$ is continuous in a closed domain D , then it must assume at least once every value between its upper and lower bounds.

(d) If $f(x, y)$ is continuous at (a, b) and $f(a, b) \neq 0$, then there exists a neighbourhood of (a, b) such that for every point of this neighbourhood $f(x, y)$ has the same sign as $f(a, b)$.

(e) If $f(x, y)$ is continuous in a closed domain D and ϵ is any positive number, then the domain D can be divided into a finite number of sub-domains such that

$$|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$$

where (x_1, y_1) and (x_2, y_2) are two points belonging to the same sub-domain.

(f) If $f(x, y)$ is continuous in a closed domain D , to every positive number ϵ corresponds another positive number δ , such that the oscillation of $f(x, y)$ is less than ϵ in any portion of D which is less than δ in all its dimensions.

(g) A function continuous in a closed domain D is uniformly continuous in the domain.

Examples.

1. Show that the following double limits exist and that the value of each is zero.

$$(i) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} xy \frac{x^2 - y^2}{x^2 + y^2}; \quad (ii) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right).$$

$$\left[(i) \left| xy \frac{x^2 - y^2}{x^2 + y^2} - 0 \right| \leq \left| xy \frac{x^2 + y^2}{x^2 + y^2} \right| \leq |x| |y| < \epsilon \right. \\ \left. \text{when } |x| < \sqrt{\epsilon}, |y| < \sqrt{\epsilon}. \right]$$

If we take $\delta = \sqrt{\epsilon}$, we get

$$\left| xy \frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \epsilon \text{ when } |x| < \delta, |y| < \delta.$$

(ii) $\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} - 0 \right| \leq |x + y|$, as $\sin \frac{1}{x}$ and $\sin \frac{1}{y}$ are each ≤ 1 ;

$$\leq |x| + |y| < \epsilon \text{ when } |x| < \frac{\epsilon}{2}, |y| < \frac{\epsilon}{2}.$$

Here $\delta = \frac{\epsilon}{2}$.

2. Show that the following double limits exist. Find the value of each.

$$(i) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 - y^2}{1 + x^2 + y^2};$$

$$(ii) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{(1 + y^2) \sin x}{x};$$

$$(iii) \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{xy}{\sqrt{(x^2 + y^2)}}.$$

[Hints. For (i) $\left| \frac{x^2 - y^2}{1 + x^2 + y^2} - 0 \right| \leq \left| \frac{x^2 + y^2}{1 + x^2 + y^2} \right|$,

Find $|x|$ and $|y|$ to make $\left| \frac{x^2 + y^2}{1 + x^2 + y^2} \right| < \epsilon$,

For (ii), use $\sin x < x$.

For (iii) note that $\frac{|x|}{\sqrt{(x^2 + y^2)}} \leq 1, \frac{|y|}{\sqrt{(x^2 + y^2)}} \leq 1$;

$$\therefore \frac{|xy|}{\sqrt{(x^2 + y^2)}} \leq \sqrt{(x^2 + y^2)}.]$$

3. Prove that

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2} \neq \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^2 - y^2}{x^2 + y^2},$$

and that $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

4. Show that for the function $f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2}$, the

repeated limits exist and are equal but the simultaneous limit does not exist when $(x, y) \rightarrow (0, 0)$.

5. Show that the following limits do not exist :

$$(i) \quad \begin{matrix} \text{Lt} \\ x \rightarrow 0 \\ y \rightarrow 0 \end{matrix} \frac{xy^3}{x^2+y^6}; \quad (ii) \quad \begin{matrix} \text{Lt} \\ x \rightarrow 0 \\ y \rightarrow 0 \end{matrix} \frac{xy^2}{x^2+y^4}.$$

[In (i) proceed along the curve $x=my^3$; in (ii) along $x=my^2$.]

6. Prove that the following functions are continuous at the origin.

$$(i) \quad f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2} \text{ when } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0.$$

$$(ii) \quad f(x, y) = x \sin \frac{1}{y} + y \sin \frac{1}{x} \text{ when } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0.$$

7. Examine the continuity of the following functions at the origin :

$$(i) \quad f(x, y) = \frac{x^3 + y^2}{x^2 + y}, \quad (x^2 + y^2 \neq 0). \quad (C. H. 1961)$$

$$(ii) \quad f(x, y) = \frac{x^2}{x^2 + y^2 - x}, \quad (x^2 + y^2 \neq 0); \quad f(0, 0) = 0. \quad (C. H. 1962)$$

8. Explain what you understand by the statement that $f(x, y, z)$ tends to a limit l as (x, y, z) tends to (a, b, c) . (C. H. 1962)

9. Prove that the function

$$f(x, y) = e^{-|x-y|/(x^2-2xy+y^2)},$$

when $(x, y) \neq (x, x)$ and $f(x, x) = 0$

is continuous at the origin.

10. (a, b) is a point in a region R in which $f(x, y)$ is continuous. If $f(a, b) \neq 0$, then prove that $f(x, y)$ has the same sign as $f(a, b)$ in some neighbourhood of (a, b) . (C. H. 1963)

[Since $f(x, y)$ is continuous at (a, b) , we have

$$|f(x, y) - f(a, b)| < \epsilon$$

when $|x - a| \leq \delta, |y - b| \leq \delta$.

$$\therefore f(a, b) - \epsilon < f(x, y) < f(a, b) + \epsilon.$$

Since $f(a, b) \neq 0$, we can take $\epsilon < |f(a, b)|$. Then $f(x, y)$ will have the same sign as $f(a, b)$ in the δ -neighbourhood of (a, b) .]

CHAPTER 17

PARTIAL DIFFERENTIATION

17.1. Partial Derivatives.

If in a function $u = f(x, y, z, \dots)$, the variables $y, z, \dots$ are kept fixed, then $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y, z, \dots) - f(x, y, z, \dots)}{\Delta x}$

if it exists, is called the *partial derivative* of u with regard to x at the point $(x, y, z, \dots)$ and is denoted by

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, D_x u, f'_x \text{ or } f_x.$$

[The first two notations were introduced by Legendre in 1786 but used by Jacobi in 1841. The notation f'_x is due to Lagrange.]

Similarly the partial derivative of u at $(x, y, z, \dots)$ with regard to y , the other variables being held constant, is denoted by

$$\frac{\partial u}{\partial y}, \frac{\partial f}{\partial y}, D_y u, f'_y \text{ or } f_y.$$

For a function $f(x, y)$ of the two variables x and y , the partial derivatives at (a, b) are defined as

$$\left(\frac{\partial f}{\partial x}\right)_{a,b} = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h},$$

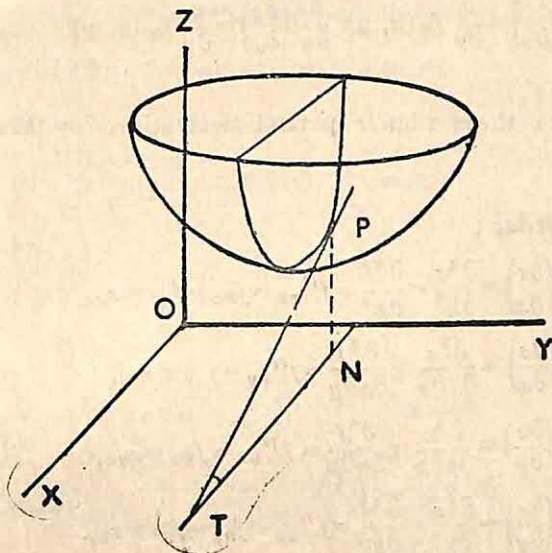
$$\left(\frac{\partial f}{\partial y}\right)_{a,b} = \lim_{k \rightarrow 0} \frac{f(a, b+k) - f(a, b)}{k}$$

if these limits exist.

Thus in forming the partial derivatives of $f(x, y)$ at (a, b) , we vary x and y along the lines $y=b$ and $x=a$ respectively and these depend upon the existence and values of $f(x, y)$ at points in the neighbourhood of (a, b) along the lines $x=a$, and $y=b$. For continuity of $f(x, y)$ at (a, b) , we have to consider the existence and values of the function at every point in the neighbourhood of (a, b) . It is therefore just possible that the partial derivatives at (a, b) exist, although the function is not continuous there.

17.2. Geometrical meaning of the partial derivatives of $f(x, y)$.

Let $z=f(x, y)$; it represents a surface geometrically. Let N



be a point (a, b) in the domain of definition of $f(x, y)$. If NP be the normal to the plane XOY through (a, b) and P is the point where NP meets the surface, its coordinates are $[a, b, f(a, b)]$. If P moves on the surface such that y remains always equal to b , then the locus of the point P is the curve of intersection of the surface and the plane $y=b$. On this curve x and z vary according to the relation $z=f(x, b)$. And so $\left(\frac{\partial z}{\partial x}\right)_{a,b}$ is the ordinary derivative of $f(x, b)$ with regard to x , for $x=a$.

It is thus the tangent of the angle which the tangent to the curve of intersection of the surface and the plane $y=b$ makes with the x -axis.

Similarly for $\left(\frac{\partial z}{\partial y}\right)_{a,b}$.

17.3. Successive Partial Derivatives of $f(x, y)$.

Consider a function $z=f(x, y)$ which admits of (first) partial derivatives

$$\frac{\partial z}{\partial x} = f_x(x, y) \quad \text{and} \quad \frac{\partial z}{\partial y} = f_y(x, y).$$

If these functions of x and y have further partial derivatives with respect to x and y , we may have partial derivatives of higher orders by repeating the process of derivation successively. Thus

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} f_x(x, y), \quad \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} f_x(x, y),$$

and so on.

We denote these higher partial derivatives by the following notations.

Second Order :

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f''_{xx} = f_{xx} = f_{x^2} = z_{xx},$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f''_{xy} = f_{xy} = z_{xy},$$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f''_{yx} = f_{yx} = z_{yx},$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f''_{yy} = f_{yy} = f_{y^2} = z_{yy}.$$

Third Order :

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x^2} \right) = \frac{\partial^3 z}{\partial x^3} = \frac{\partial^3 f}{\partial x^3} = f_{xxx} = f_{x^3} = z_{xxx},$$

$$\frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial x^2} \right) = \frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x^2 \partial y} = f_{x^2 y} = z_{x^2 y},$$

and so on.

[Observe. $\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$ is represented by $\frac{\partial^2 z}{\partial x \partial y}$ by French writers and their followers. The same is represented by $\frac{\partial^2 z}{\partial y \partial x}$ by English writers and their followers. In this book we shall follow the first method.]

174. Order of successive derivation.

The second and higher order derivatives involve repeated limiting processes. $\frac{\partial^2 z}{\partial x \partial y}$, for example, is obtained by first forming the derivative of z with respect to x —a limiting process, and then

forming the derivative of $\frac{\partial z}{\partial x}$ with regard to y —another limiting process. An inversion of the order of procedure gives us the derivative $\frac{\partial^2 z}{\partial y \partial x}$; and these being repeated limits there is no reason to suppose that the two will always be the same.

An example to illustrate this is the following :

$$\text{If } f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2} \text{ (} x, y \text{ both not zero),}$$

$$f(0, 0) = 0,$$

$$\text{then at } x = y = 0, \frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}.$$

$$\begin{aligned} \text{For } \left(\frac{\partial f}{\partial x} \right)_{(0, y)} &= \lim_{h \rightarrow 0} \frac{f(h, y) - f(0, y)}{h} = \lim_{h \rightarrow 0} \frac{hy \frac{h^2 - y^2}{h^2 + y^2} - 0}{h} \\ &= -y, \end{aligned}$$

$$\begin{aligned} \text{so that } \left(\frac{\partial^2 f}{\partial x \partial y} \right)_{(0, 0)} &= \lim_{k \rightarrow 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \\ &= \lim_{k \rightarrow 0} \frac{-k}{k} = -1. \end{aligned}$$

$$\text{Again } \left(\frac{\partial f}{\partial y} \right)_{(x, 0)} = \lim_{k \rightarrow 0} \frac{xk \frac{x^2 - k^2}{x^2 + k^2} - 0}{k} = x,$$

$$\text{whence } \left(\frac{\partial^2 f}{\partial y \partial x} \right)_{(0, 0)} = \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

$$\text{Thus at } x = y = 0, \frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}.$$

17.5. Inversion of derivation.

The following theorem gives a sufficient condition for the equality of the two derivatives.

Theorem : If $f(x, y)$ be a function of the two variables x and y defined in a domain D , such that f''_{ux} and f''_{xy} exist and are continuous in D , then $f''_{ux} = f''_{xy}$ in D .

Let $U = f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)$;
also let

$$F(v) = f(x+h, v) - f(x, v),$$

$$\text{and } G(u) = f(u, y+k) - f(u, y).$$

Then $U = F(y+k) - F(y) = kF'_v(y+\theta'k)$, by the mean value theorem,

$$= k\{f'_v(x+h, y+\theta'k) - f'_v(x, y+\theta'k)\}$$

$$= hkf''_{vx}(x+\theta h, y+\theta'k), \text{ by the mean value theorem,}$$

where both θ and θ' lie between 0 and 1.

Again $U = G(x+h) - G(x) = hG'_x(x+\theta_1h)$ by the mean value theorem,

$$= h\{f'_x(x+\theta_1h, y+k) - f'_x(x+\theta_1h, y)\}$$

$$= hkf''_{xy}(x+\theta_1h, y+\theta'_1k), \text{ by the mean value theorem}$$

where both θ_1 and θ'_1 lie between 0 and 1.

Equating the two values of U and dividing by hk , we have

$$f''_{vx}(x+\theta h, y+\theta'k) = f''_{xy}(x+\theta_1h, y+\theta'_1k).$$

Since the derivatives f''_{vx} and f''_{xy} are supposed continuous, the two members of the equation approach f''_{vx} and f''_{xy} respectively as h and k approach zero, and the theorem is proved.

17.6. Theorem of Schwarz.

The following more general theorem giving sufficient conditions for the equality of the two derivatives is due to Schwarz.

Theorem : If (x, y) be a point in the domain of definition of $f(x, y)$ such that (i) $f_x(x, y)$ and $f_y(x, y)$ exist in a neighbourhood of (x, y) , and (ii) $f_{yx}(x, y)$ exists and is continuous at (x, y) , then $f_{xy}(x, y)$ exists and is equal to $f_{yx}(x, y)$.

Consider the expression

$$U = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y).$$

$$\text{Let } \phi(y) = f(x+h, y) - f(x, y).$$

$$\text{Then } U = \phi(y+k) - \phi(y).$$

$$\dots \dots \dots (1)$$

Since $f_y(x, y)$ exists in a neighbourhood of (x, y) , the function $\phi(y)$ is derivable in $(y, y+k)$ and so applying the mean value theorem to (1), we get

$$U = k\phi_y(y+\theta k)$$

$$= k[f_y(x+h, y+\theta k) - f_y(x, y+\theta k)]. \dots \dots (2)$$

Again $f_{yx}(x, y)$ exists in a neighbourhood of (x, y) , the function $f_y(x, y+\theta k)$ is therefore derivable with respect to x in $(x, x+h)$; so applying the mean value theorem to (2), we have

$$U = hf_{yx}(x + \theta'h, y + \theta h).$$

$$\begin{aligned} \therefore (h, k) \xrightarrow{\text{Lt}} (0, 0) f_{yx}(x + \theta'h, y + \theta k) &= (h, k) \xrightarrow{\text{Lt}} (0, 0) \left(\frac{U}{hk} \right) \\ &= \lim_{k \rightarrow 0} \frac{1}{k} \left[\lim_{h \rightarrow 0} \frac{f(x+h, y+k) - f(x, y+k)}{h} \right. \\ &\quad \left. - \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \right]. \end{aligned}$$

Since $f_{yx}(x, y)$ exists, this is equal to

$$\lim_{k \rightarrow 0} \frac{f_{yx}(x, y+k) - f_{yx}(x, y)}{k}.$$

$$\therefore f_{yx}(x, y) = \lim_{k \rightarrow 0} \frac{f_{yx}(x, y+k) - f_{yx}(x, y)}{k} = f_{xy}(x, y),$$

since $f_{yx}(x, y)$ exists and is continuous.

17.7. Boundedness of the partial derivatives implies continuity of the function.

It has been remarked in Art. 17.1 that the partial derivatives of a function $f(x, y)$ may exist at a point where the function is not continuous. If however the partial derivatives are bounded, the function must be continuous; and we have the following theorem.

Theorem : If a function $f(x, y)$ defined in a domain D possesses partial derivatives f_x and f_y in D and they are bounded in the domain, then $f(x, y)$ is continuous in D .

Let (x, y) and $(x+h, y+k)$ be two points in the domain D such that the line joining them lies wholly in the domain. We have

$$\begin{aligned} f(x+h, y+k) - f(x, y) &= [f(x+h, y+k) - f(x, y+k)] \\ &\quad + [f(x, y+k) - f(x, y)]. \end{aligned} \quad \dots (1)$$

Since f_x and f_y are bounded in D , they are finite in D and so $f(x, y)$ is derivable with respect to both x and y . Hence applying the mean value theorem to (1), we get

$$f(x+h, y+k) - f(x, y) = hf_x(x + \theta h, y+k) + kf_y(x, y + \theta'k) \quad \dots (2)$$

where both θ and θ' lie between 0 and 1.

Since f_x and f_y are bounded in D , we may take

$$|f_x| \leq M, \quad |f_y| \leq M$$

where M is a constant. Thus we have

$$|f(x+h, y+k) - f(x, y)| \leq |hf_x(x+\theta h, y+k)| \\ + |kf_y(x, y+\theta'k)| \\ \leq M(|h| + |k|).$$

If h and k are arbitrarily small, we may put $|h| + |k| < \frac{\epsilon}{M}$,

and $\therefore |f(x+h, y+k) - f(x, y)| < \epsilon$, proving the continuity of $f(x, y)$ at the point (x, y) in D .

Cor. If f_x and f_y are continuous at (x, y) , then $f(x, y)$ is also continuous at (x, y) .

Examples A.

1. Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^3}.$$

[Hint. $x^3 + y^3 + z^3 - 3xyz$
 $= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z).$]

2. If $f(x, y, z)$ satisfies the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

then $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ and $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}$ all satisfy the same equation.

3. If $u = f(ax^2 + 2hxy + by^2)$ and $v = \phi(ax^2 + 2hxy + by^2)$, prove that
- $$\frac{\partial}{\partial y} \left(u \frac{\partial v}{\partial x} \right) = \frac{\partial}{\partial x} \left(u \frac{\partial v}{\partial y} \right).$$

4. If $\theta = t^n e^{-\frac{r^2}{4t}}$, find n which will make

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) = \frac{\partial \theta}{\partial t}.$$

$$\left[n = -\frac{3}{2} \right]$$

5. If $x = r \cos \theta$, $y = r \sin \theta$, account for the results

$$\frac{\partial r}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial r}} \quad \text{and} \quad \frac{\partial \theta}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial \theta}}.$$

(C. H. 1964)

[See Hardy—Pure Mathematics, Examples LX, 2.]

6. (i) If $\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$, prove that

$$\Delta(uv) = u \Delta v + v \Delta u + 2 \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right).$$

(ii) If $\phi(x, y, z)$ be a function such that $\Delta\phi = 0$

and
$$x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} + z \frac{\partial \phi}{\partial z} = n\phi(x, y, z),$$

apply the above result to prove

$$\Delta(r^m\phi) = r^{m-2} [m(m+1) + 2mn] \phi(x, y, z),$$

where $r^2 = x^2 + y^2 + z^2$ and m, n are positive integers. (C. H. 1960)

[(i) We have $\frac{\partial}{\partial x}(uv) = u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x}$,

$$\therefore \frac{\partial^2}{\partial x^2}(uv) = u \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}.$$

Similarly $\frac{\partial^2}{\partial y^2}(uv) = u \frac{\partial^2 v}{\partial y^2} + v \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y},$

$$\frac{\partial^2}{\partial z^2}(uv) = u \frac{\partial^2 v}{\partial z^2} + v \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial u}{\partial z} \frac{\partial v}{\partial z}.$$

Adding,

$$\Delta(uv) = u \Delta v + v \Delta u + 2 \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right).$$

(ii) By applying the above result

$$\Delta(r^m\phi) = r^m \Delta\phi + \phi \Delta r^m + 2 \left(\frac{\partial r^m}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial r^m}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial r^m}{\partial z} \frac{\partial \phi}{\partial z} \right).$$

Now $\frac{\partial r^m}{\partial x} = m r^{m-1} \frac{\partial r}{\partial x} = m r^{m-1} \cdot \frac{x}{r} = m r^{m-2} \cdot x,$

and
$$\begin{aligned} \frac{\partial^2(r^m)}{\partial x^2} &= \frac{\partial}{\partial x}(m r^{m-2} \cdot x) = m r^{m-2} + m(m-2) r^{m-3} \cdot \frac{x^2}{r} \\ &= m r^{m-2} + m(m-2) r^{m-4} \cdot x^2. \end{aligned}$$

$$\therefore \Delta(r^m) = 3m r^{m-2} + m(m-2) r^{m-2} = m(m+1) r^{m-2}.$$

Again $\sum \frac{\partial r^m}{\partial x} \cdot \frac{\partial \phi}{\partial x} = m r^{m-2} \sum x \frac{\partial \phi}{\partial x} = mn r^{m-2} \phi(x, y, z).$

Substituting values we get

$$\Delta(r^m\phi) = r^{m-2} [m(m+1) + 2mn] \phi(x, y, z).]$$

7. If $u(x, y, z, t) = \frac{1}{r}f(t+r) + \frac{1}{r}g(t-r)$, where $r^2 = x^2 + y^2 + z^2$, prove that $u_{xx} + u_{yy} + u_{zz} = u_{tt}$. (C. H. 1961)

[From $r^2 = x^2 + y^2 + z^2$, $r_x = \frac{x}{r}$, $r_y = \frac{y}{r}$, $r_z = \frac{z}{r}$.

Also from $u = \frac{f(t+r)}{r} + \frac{g(t-r)}{r}$, i.e., $ru = f(t+r) + g(t-r)$, we get

$$r_x u + r u_x = \{f'(t+r) - g'(t-r)\} \cdot r_x = (f' - g') \cdot \frac{x}{r},$$

$$\therefore r u_x = (f' - g' - u) r_x, \text{ or, } r^2 u_x = (f' - g' - u)x. \quad \dots \quad (1)$$

Differentiating with respect to x ,

$$2r r_x u_x + r^2 u_{xx} = f' - g' - u + \frac{x^2}{r} (f'' + g'') - 3x u_x,$$

$$\text{i.e., } r^2 u_{xx} = f' - g' - u + \frac{x^2}{r} (f'' + g'') - 3x u_x.$$

$$\text{Hence } r^2 (u_{xx} + u_{yy} + u_{zz}) = 3(f' - g' - u) + r(f'' + g'') - 3(xu_x + yu_y + zu_z). \quad \dots \quad (2)$$

$$\text{Now } xu_x = (f' - g' - u) \cdot \frac{x^2}{r^2}; \quad \therefore \Sigma xu_x = f' - g' - u. \quad \dots \quad (3)$$

$$\text{Again from } ru = f(t+r) + g(t-r),$$

$$ru_t = f' + g', \quad ru_{tt} = f'' + g''.$$

Hence substituting in (2),

$$r^2 \Sigma u_{xx} = 3(f' - g' - u) + r^2 u_{tt} - 3(f' - g' - u),$$

$$\text{i.e., } u_{xx} + u_{yy} + u_{zz} = u_{tt}. \quad]$$

8. If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, where u is a function of x, y, z , prove that $(u_x)^2 + (u_y)^2 + (u_z)^2 = 2(xu_x + yu_y + zu_z)$.

9. Show that for $f(x, y) = |x^2 - y^2|$, $f_{xy}(0, 0) = f_{yx}(0, 0)$. (C. H. 1966)

10. For the function

$$f(x, y) = \sqrt{(x^2 + y^2)} \sin 2\phi \text{ where } \phi = \tan^{-1} \frac{y}{x},$$

$$\text{and } f(0, 0) = 0,$$

show that $f_{xy}(0, 0) = f_{yx}(0, 0)$.

11. For the function

$$f(x, y) = xy \frac{x^2 + a^2 y^2}{x^2 + y^2}, \quad (x^2 + y^2 \neq 0, \text{ and } a \neq 1)$$

$$\text{and } f(0, 0) = 0,$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. (C. H. 1960)

12. Examine whether

$$f(x, y) = \begin{cases} xy & \text{if } |x| \geq |y|, \\ -xy & \text{if } |x| < |y| \end{cases}$$

is continuous at the origin.

Is $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for this function at the origin? Give reasons for your answer. (C. H. 1963)

13. For $f(x, y) = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$ if $xy \neq 0$, and is 0 elsewhere, show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

$$14. \text{ Let } f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq 0 \\ = 0 \text{ otherwise.}$$

Prove that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but $f(x, y)$ is discontinuous at $(0, 0)$. (C. H. 1966)

$$\left[f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0. \right.$$

Similarly $f_y(0, 0) = 0$.

To prove the discontinuity at $(0, 0)$, let (x, y) approach $(0, 0)$ along $y = mx$.]

$$15. \text{ Given } f(x, y) = \frac{2xy}{x^2 + y^2}, \quad (x^2 + y^2 \neq 0)$$

$$f(0, 0) = 0,$$

prove that the function has partial derivatives everywhere. Is it continuous everywhere? (C. H. 1962)

$$16. \text{ If } f(x, y) = \frac{x^3 + y^3}{x - y}, \text{ when } x \neq y,$$

$$f(x, y) = 0, \quad \text{when } x = y$$

show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but $f(x, y)$ is discontinuous at the origin.

[To prove the discontinuity at the origin, put $y = x - mx^3$.]

17.8. Total Differential.

Consider the function $u = f(x, y)$ of the two variables x and y and let x and y be given increments Δx and Δy . Let us put further

$$\rho = |\Delta x| + |\Delta y|, \text{ or else } \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

The function is said to be *differentiable* at the point (x, y) if it is well determined in the neighbourhood of this point and if its increment Δu , corresponding to the increments Δx and Δy of x and y , can be expressed in the form

$$\Delta u = A\Delta x + B\Delta y + \epsilon\rho, \quad \dots \dots (1)$$

A, B being independent of Δx and Δy and ϵ tending to zero with ρ . For $\epsilon\rho$ we may as well write $\epsilon_1\Delta x + \epsilon_2\Delta y$ where ϵ_1 and ϵ_2 both tend to zero with Δx and Δy .

When u is differentiable, the part of Δu which is linear in Δx and Δy , that is, the part $A\Delta x + B\Delta y$, is called the *total differential* of u and represented by du . We have therefore by definition

$$du = A\Delta x + B\Delta y.$$

On putting $\Delta y = 0$ in (1) and making Δx tend to zero, we get

$$\text{Lt } \frac{du}{\Delta x} = \frac{\partial u}{\partial x} = A; \text{ similarly } \frac{\partial u}{\partial y} = B.$$

$$\text{Consequently } du = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \quad \dots \dots (2)$$

The *partial differentials* of u are defined by

$$d_x u = \frac{\partial u}{\partial x} \Delta x, \quad d_y u = \frac{\partial u}{\partial y} \Delta y,$$

and are thus the products of the partial derivatives of u with the arbitrary increments Δx and Δy of the corresponding variables x and y . It is clear from (2) that the total differential of u is the sum of its partial differentials.

Putting $u = f(x, y) = x$, we have $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0$ and so from (2) we get $dx = \Delta x$; similarly by putting $u = y$, we get $dy = \Delta y$. Hence (2) may be written as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

When the number of variables is greater than two, we shall similarly have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz + \dots$$

Note. A function having a total differential is said to be *differentiable*.

Differentials of higher orders

From $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$, we have

$$\begin{aligned} d^2u &= d(du) = d\left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) \\ &= d\left(\frac{\partial u}{\partial x}\right)dx + \frac{\partial u}{\partial x} d(dx) + d\left(\frac{\partial u}{\partial y}\right)dy + \frac{\partial u}{\partial y} d(dy) \\ &= \frac{\partial^2 u}{\partial x^2} (dx)^2 + \frac{\partial^2 u}{\partial x \partial y} dx dy + \frac{\partial^2 u}{\partial y \partial x} dy dx + \frac{\partial^2 u}{\partial y^2} (dy)^2 \\ &\quad + \frac{\partial^2 u}{\partial y \partial x} dx dy + \frac{\partial^2 u}{\partial y^2} (dy)^2 + \frac{\partial^2 u}{\partial y^2} dy^2. \end{aligned}$$

Writing $(dx)^2 = dx^2$, $(dy)^2 = dy^2$ and noting that if x and y are independent variables, dx , dy are constants so that $d(dx) = d^2x = 0$, $d(dy) = d^2y = 0$, we get

$$d^2u = \frac{\partial^2 u}{\partial x^2} dx^2 + 2 \frac{\partial^2 u}{\partial x \partial y} dx dy + \frac{\partial^2 u}{\partial y^2} dy^2.$$

This is usually written as

$$d^2u = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^{(2)} u.$$

Similarly $d^r u = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^{(r)} u.$

For a function of more than two variables

$$d^2u = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} + \dots\right)^{(2)} u, \text{ etc.}$$

17.9. A sufficient condition for differentiability.

Theorem. If (x, y) be a point in the domain of definition of $f(x, y)$ and if (i) f_x and f_y exist at (x, y) and are both finite, (ii) one

at least of the partial derivatives, say f_y , exists everywhere in a certain neighbourhood of (x, y) and is further continuous at (x, y) , then $f(x, y)$ is differentiable at (x, y) .

By hypothesis f_y exists in a certain neighbourhood of (x, y) . Let $(x+h, y+k)$ be a point of this neighbourhood. We have

$$f(x+h, y+k) - f(x, y) = f(x+h, y+k) - f(x+h, y) + f(x+h, y) - f(x, y) \quad \dots (1)$$

f_y being continuous at (x, y) it exists finitely in the neighbourhood and so the function $f(x+h, y)$ is derivable with respect to y in the interval $(y, y+k)$; therefore by the mean value theorem

$$f(x+h, y+k) - f(x+h, y) = kf_y(x+h, y+\theta k)$$

where θ lies between 0 and 1 and depends upon h and k .

If σ be a function of h and k so that $\sigma \rightarrow 0$ as $(h, k) \rightarrow 0$, we can write $kf_y(x+h, y+\theta k) = kf_y(x, y) + k\sigma$... (2)

Again because f_x exists finitely, we have when $h \rightarrow 0$,

$$\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x(x, y);$$

and so we can write

$$f(x+h, y) - f(x, y) = hf_x(x, y) + h\rho \quad \dots (3)$$

where $\rho \rightarrow 0$ with h .

$\therefore$ from (1), (2), (3), we get

$$f(x+h, y+k) - f(x, y) = hf_x(x, y) + kf_y(x, y) + h\rho + k\sigma.$$

showing that $f(x, y)$ is differentiable at (x, y) .

Cor. 1. If $f(x, y)$ is differentiable at (x, y) , it is necessarily continuous at (x, y) .

For as $(h, k) \rightarrow 0$, $[f(x+h, y+k) - f(x, y)] \rightarrow 0$;

$$\therefore \lim_{(h, k) \rightarrow (0, 0)} f(x+h, y+k) = f(x, y).$$

The converse of this result is not true.

Cor. 2. If $f(x, y)$ is differentiable at (x, y) then it also necessarily possesses the partial derivatives $f_x(x, y)$, $f_y(x, y)$ at (x, y) .

Putting $k=0$ and making $h \rightarrow 0$, we have from (3)

$$\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x(x, y).$$

Similarly putting $h=0$ and making $k \rightarrow 0$, we get from (3)

$$\lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k} = f_y(x, y).$$

Here again the converse is not true.

17'10. Differentiation in a given direction. Space derivatives.

An important property of differentiable functions is that they not only possess partial derivatives with respect to x and y , or, as we also say, in the x - and y - directions, but they also have derivatives in any other direction. By derivative in the direction α , we mean the following.

We let the point $(x+h, y+k)$ approach the point (x, y) so that it is always on the line through (x, y) which makes the constant angle α with the positive direction of the x -axis. In other words, h and k do not tend to zero independently of one another, but satisfy the relations

$$h = \rho \cos \alpha, \quad k = \rho \sin \alpha,$$

where ρ is the distance $\sqrt{h^2 + k^2}$ of the point $(x+h, y+k)$ from the point (x, y) and tends to zero as h and k do. If we now form the difference $f(x+h, y+k) - f(x, y)$ and divide by ρ , we call the limit of the expression

$$D_{(\alpha)} f(x, y) = \lim_{\rho \rightarrow 0} \frac{f(x + \rho \cos \alpha, y + \rho \sin \alpha) - f(x, y)}{\rho}$$

the derivative of the function $f(x, y)$ at the point (x, y) in the direction α , provided the limit exists. $\alpha=0$ gives the derivative $\frac{\partial f}{\partial x}$ and $\alpha=\frac{\pi}{2}$, the derivative $\frac{\partial f}{\partial y}$.

Now if $f(x, y)$ is differentiable, we have

$$\begin{aligned} f(x+h, y+k) - f(x, y) &= h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} + \epsilon \rho \\ &= \rho \left(\frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha + \epsilon \right). \end{aligned}$$

As $\rho \rightarrow 0$, so does ϵ and for the derivative in the direction α , we obtain the expression

$$D_{(\alpha)} f(x, y) = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha;$$

it is thus a linear function of the derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in the x - and y -directions, with the coefficients $\cos \alpha$, $\sin \alpha$. The result always holds good provided $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous.

In the same way, a differentiable function $f(x, y, z)$ of three variables can be differentiated in any direction. If α, β, γ be the direction angles and $h = \rho \cos \alpha$, $k = \rho \cos \beta$, $l = \rho \cos \gamma$, then just as above we get the expression

$$\frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \cos \beta + \frac{\partial f}{\partial z} \cos \gamma$$

for the derivative in the direction (α, β, γ) .

If the direction η has direction cosines (l, m, n) , the *space derivative* in the direction η is

$$\frac{\partial f}{\partial \eta} = l \frac{\partial f}{\partial x} + m \frac{\partial f}{\partial y} + n \frac{\partial f}{\partial z}.$$

Observation :

- (1) $f(x, y)$ possesses the directional derivative $D_{(\alpha)} f(x, y)$ if

(C. H. 1969)

$$\lim_{\rho \rightarrow 0} \frac{f(x + \rho \cos \alpha, y + \rho \sin \alpha) - f(x, y)}{\rho}$$

exists whether $f(x, y)$ be differentiable or not. If however $f(x, y)$ be differentiable, $D_{(\alpha)} f(x, y)$ can be exhibited in the simple form

$$\frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha.$$

- (2) The derivatives f_x and f_y in the x - and y -directions are partial.

(3) A function of more than one variable is *derivable* if the directional derivatives as well as the partial derivatives exist whether finite or not. A necessary condition for the *differentiability* of such a function is that all directional derivatives as also the partial derivatives should exist finitely.

17.11. Function of functions.

Let $x = f_1(\xi, \eta)$ and $y = f_2(\xi, \eta)$ be two functions having continuous partial derivatives with regard to ξ and η and let $u = F(x, y)$

be a function of x and y having continuous partial derivatives with respect to x and y , u is then a function ξ, η . For

$$u = F[f_1(\xi, \eta), f_2(\xi, \eta)] = \phi(\xi, \eta).$$

Theorem : The function $u = \phi(\xi, \eta)$ admits of partial derivatives with regard to ξ and η and the derivatives are given by

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi},$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}.$$

Let ξ receive an increment $\Delta \xi$, then corresponding to it, x and y will receive increments Δx and Δy where

$$\Delta x = f_1(\xi + \Delta \xi, \eta) - f_1(\xi, \eta),$$

$$\Delta y = f_2(\xi, \eta + \Delta \eta) - f_2(\xi, \eta).$$

Since the partial derivatives of $f_1(\xi, \eta)$ are supposed to exist, we have, $\text{Lt}_{\Delta \xi \rightarrow 0} \frac{\Delta x}{\Delta \xi} = \text{Lt}_{\Delta \xi \rightarrow 0} \frac{f_1(\xi + \Delta \xi, \eta) - f_1(\xi, \eta)}{\Delta \xi} = \frac{\partial x}{\partial \xi}$,

and similarly $\text{Lt}_{\Delta \xi \rightarrow 0} \frac{\Delta y}{\Delta \xi} = \frac{\partial y}{\partial \xi}$.

Again as Δx and Δy are increments of x and y corresponding to the increment $\Delta \xi$ of ξ , u will receive an increment Δu given by

$$\begin{aligned} \Delta u &= F(x + \Delta x, y + \Delta y) - F(x, y) \\ &= F(x + \Delta x, y + \Delta y) - F(x, y + \Delta y) + F(x, y + \Delta y) - F(x, y), \\ &= F_x(x + \theta_1 \Delta x, y + \Delta y) \cdot \Delta x + F_y(x, y + \theta_2 \Delta y) \Delta y, \dots \quad (1) \end{aligned}$$

by the mean value theorem, where $0 < \theta_1, \theta_2 < 1$.

$$\therefore \frac{\Delta u}{\Delta \xi} = F_x(x + \theta_1 \Delta x, y + \Delta y) \frac{\Delta x}{\Delta \xi} + F_y(x, y + \theta_2 \Delta y) \frac{\Delta y}{\Delta \xi}, \dots (2)$$

Now when $\Delta \xi \rightarrow 0$, Δx and Δy both $\rightarrow 0$, so that

$$F_x(x + \theta_1 \Delta x, y + \Delta y) \rightarrow F_x(x, y),$$

$$\text{and } F_y(x, y + \theta_2 \Delta y) \rightarrow F_y(x, y),$$

F_x and F_y being continuous. Hence proceeding to the limit when $\Delta \xi \rightarrow 0$, we have from (2),

$$\frac{\partial u}{\partial \xi} = F_x(x, y) \frac{\partial x}{\partial \xi} + F_y(x, y) \frac{\partial y}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi}.$$

Similarly $\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}.$

More generally, if $u = F(x, y, z, \dots)$ and

$$x = f_1(\xi, \eta, \zeta, \dots), y = f_2(\xi, \eta, \zeta, \dots), z = f_3(\xi, \eta, \zeta, \dots) \dots$$

then

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \xi} + \dots,$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \eta} + \dots,$$

$$\frac{\partial u}{\partial \zeta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \zeta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \zeta} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \zeta} + \dots$$

... ..

17.12. Theorem of the total differential co-efficient.

Let $f(x, y)$ be a function of the two variables x and y having continuous partial derivatives. Suppose further that x and y are both functions of a third variable t , that is $x = u(t)$, $y = v(t)$, both possessing continuous differential coefficients $\frac{dx}{dt}$, $\frac{dy}{dt}$ with respect to

t . Then $f(x, y) = f\{u(t), v(t)\}$ and is thus a composite function of t —say $F(t)$. We proceed to find $F'(t)$.

Let Δt be an increment of t , corresponding to which the increments of x and y are Δx and Δy respectively. Then

$$\begin{aligned} F'(t) &= \lim_{\Delta t \rightarrow 0} \frac{f\{u(t + \Delta t), v(t + \Delta t)\} - f\{u(t), v(t)\}}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left[\frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} \cdot \frac{\Delta x}{\Delta t} \right. \\ &\quad \left. + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \cdot \frac{\Delta y}{\Delta t} \right] \\ &= \lim_{\Delta t \rightarrow 0} \left[f_x(x + \theta \Delta x, y + \Delta y) \cdot \frac{\Delta x}{\Delta t} \right. \\ &\quad \left. + f_y(x, y + \theta' \Delta y) \cdot \frac{\Delta y}{\Delta t} \right], \quad (0 < \theta, \theta' < 1). \end{aligned}$$

As $\Delta t \rightarrow 0$, Δx , Δy both tend to 0 and

$$\frac{\Delta x}{\Delta t} \rightarrow \frac{dx}{dt}, \quad \frac{\Delta y}{\Delta t} \rightarrow \frac{dy}{dt}.$$

Also $f_x(x+\theta\Delta x, y+\Delta y) \rightarrow f_x(x, y)$ and $f_y(x, y+\theta'\Delta y) \rightarrow f_y(x, y)$ as these derivatives are supposed to be continuous. Hence

$$F'(t) = f_x(x, y) \frac{dx}{dt} + f_y(x, y) \frac{dy}{dt},$$

$$\text{i.e., } \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

More generally for a function of several variables $f(x, y, z, \dots)$,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} + \dots,$$

$x, y, z, \dots$ being all functions of t .

Note 1. This result may be considered as a generalisation of the ordinary rules for differentiation of the sum, product, etc., of several functions of a single variable.

(i) If $y = u + v + w$,

$$\frac{dy}{dx} = \frac{\partial y}{\partial u} \frac{du}{dx} + \frac{\partial y}{\partial v} \frac{dv}{dx} + \frac{\partial y}{\partial w} \frac{dw}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}.$$

(ii) If $y = uv$,

$$\frac{dy}{dx} = \frac{\partial y}{\partial u} \frac{du}{dx} + \frac{\partial y}{\partial v} \frac{dv}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}.$$

(iii) If $y = \frac{u}{v} = uv^{-1}$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\partial y}{\partial u} \frac{du}{dx} + \frac{\partial y}{\partial v} \frac{dv}{dx} = v^{-1} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx} \\ &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}. \end{aligned}$$

Note 2. If in $u = f(x, y)$, y be a function of x , then u is a function of x .

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x} \frac{dx}{dx} + \frac{\partial u}{\partial y} \frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}.$$

In particular if $f(x, y) = 0$, then $\frac{du}{dx} = 0$ and we get

$$\frac{dy}{dx} = - \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

17.13. Homogeneous functions.

Definition (1). If the function $f(x, y, z, \dots)$ be such that a substitution $tx, ty, tz, \dots$ for $x, y, z, \dots$, has the effect of multiplying the original function by t^n , i.e., if $f(tx, ty, tz, \dots) = t^n f(x, y, z, \dots)$, then $f(x, y, z, \dots)$ is a homogeneous function in $x, y, z, \dots$ of degree n .

Definition (2). Putting $t = \frac{1}{x}$ in the above, we find

$$x^n f\left(1, \frac{y}{x}, \frac{z}{x}, \dots\right) = f(x, y, z, \dots)$$

Thus $f(x, y, z, \dots)$ is a homogeneous function in $x, y, z, \dots$ of degree n , if it is expressible as $x^n f\left(\frac{y}{x}, \frac{z}{x}, \dots\right)$, x being any of the variables occurring in the function.

17.14. Euler's theorem.

If $f(x, y, z, \dots)$ be a homogeneous function in $x, y, z, \dots$ of degree n , having continuous partial derivatives, then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} + \dots = n f(x, y, z, \dots).$$

As $f(x, y, z, \dots)$ is a homogeneous function of degree n ,

$$f(tx, ty, tz, \dots) = t^n f(x, y, z, \dots).$$

Differentiating both sides with respect to t , we have

$$\frac{\partial f}{\partial (tx)} \cdot x + \frac{\partial f}{\partial (ty)} \cdot y + \dots = n t^{n-1} f(x, y, z, \dots).$$

Putting $t=1$ in this, we get

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + \dots = n f(x, y, z, \dots).$$

Cor. 1. Differentiating this again with respect to t , we get

$$\begin{aligned} \left(x \frac{\partial}{\partial (tx)} + y \frac{\partial}{\partial (ty)} + \dots \right) \left(x \frac{\partial f}{\partial (tx)} + y \frac{\partial f}{\partial (ty)} + \dots \right) \\ = n(n-1) t^{n-2} f(x, y, z, \dots). \end{aligned}$$

Putting $t=1$,

$$\begin{aligned} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \dots \right)^{(2)} f(x, y, z, \dots) \\ = n(n-1) f(x, y, z, \dots). \end{aligned}$$

More generally

$$\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \dots \right)^{(r)} f(x, y, z, \dots) \\ = n(n-1) \dots (n-r+1) f(x, y, z, \dots).$$

Cor. 2. Let $u=f(H_n)$, where H_n is a homogeneous function of degree n . Suppose we obtain from this the relation $H_n=F(u)$. Then

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + \dots = nH_n = nF(u),$$

$$\text{or,} \quad F'(u) \left\{ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \dots \right\} = nF(u),$$

$$\text{i.e.,} \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \dots = n \frac{F(u)}{F'(u)}.$$

17.15. Converse to Euler's theorem.

If $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf(x, y, z)$, to prove that $f(x, y, z)$ is homogeneous and of degree n .

Replacing the variables x, y, z by ξ, η, ζ , we are given

$$\xi \frac{\partial f}{\partial \xi} + \eta \frac{\partial f}{\partial \eta} + \zeta \frac{\partial f}{\partial \zeta} = nf(\xi, \eta, \zeta).$$

Now putting $\xi=tx, \eta=ty, \zeta=tz$, we get

$$x \frac{\partial f}{\partial \xi} + y \frac{\partial f}{\partial \eta} + z \frac{\partial f}{\partial \zeta} = \frac{n}{t} f(tx, ty, tz). \quad \dots (1)$$

Let $t^n f(x, y, z) - f(tx, ty, tz) = \phi(t)$; then

$$\begin{aligned} \phi'(t) &= nt^{n-1} f(x, y, z) - \left(x \frac{\partial f}{\partial \xi} + y \frac{\partial f}{\partial \eta} + z \frac{\partial f}{\partial \zeta} \right) \\ &= nt^{n-1} f(x, y, z) - \frac{n}{t} f(tx, ty, tz), \dots \text{by (1)} \\ &= \frac{n}{t} \phi(t). \end{aligned}$$

This gives $\frac{\phi'(t)}{\phi(t)} = \frac{n}{t}$.

Integrating $\phi(t) = ct^n$. But $\phi(1) = 0$, giving $c = 0$.

$$\therefore \phi(t) = 0,$$

$$\text{i.e.,} \quad f(tx, ty, tz) = t^n f(x, y, z)$$

showing that $f(x, y, z)$ is homogeneous and of degree n .

[See Ex. 18 for an alternative treatment.]

Examples B

1. Show that the function

$$f(x, y) = |x| + |y|$$

is continuous but not differentiable at $(0, 0)$.

(C. H. 1965)

$$[\text{Since } f(0, 0) = 0, \quad |f(x, y) - f(0, 0)|$$

$$= |x| + |y| < \epsilon \text{ if } |x| < \frac{1}{2}\epsilon, |y| < \frac{1}{2}\epsilon.$$

$$\text{Now } f_x(x, 0) \text{ at } (0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{|h|}{h} = +1 \text{ or } -1 \text{ acc. as } h \geq 0.$$

As $f_x(0, 0)$ and similarly $f_y(0, 0)$ do not exist, $f(x, y)$ is not differentiable.]

2. Examine whether
- $f(x, y) = \sqrt{|xy|}$
- is totally differentiable at the origin.

(C. H. 1963)

[Δf at the origin $= f(h, k) - f(0, 0) = \sqrt{|hk|}$. To be differentiable at the origin, Δf should be expressible in the form

$$\Delta f = [hf_x + kf_y + h\rho + k\sigma]_{x=0, y=0}.$$

$$\text{Now } f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

$$f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0,$$

$\therefore \Delta f$ at the origin $= \sqrt{|hk|}$ should be equal to $h\rho + k\sigma$.

We shall prove that $\sqrt{|hk|} \neq h\rho + k\sigma$ as h, k tend to 0 in any manner.

Put $h = r \cos \theta$, $k = r \sin \theta$, then we have to show that as $r \rightarrow 0$,

$$r \sqrt{|\cos \theta \sin \theta|} \neq r(\rho \cos \theta + \sigma \sin \theta)$$

$$\text{i.e., } \sqrt{|\cos \theta \sin \theta|} \neq \rho \cos \theta + \sigma \sin \theta.$$

Now as $r \rightarrow 0$, ρ and σ tend to zero and so $\rho \cos \theta + \sigma \sin \theta \rightarrow 0$; but $\sqrt{|\cos \theta \sin \theta|}$ does not tend to zero. So $\sqrt{|xy|}$ is not differentiable at the origin.]

3. Examine whether

$$f(x, y) = \frac{xy^2}{x^2 + y^2} \quad (x^2 + y^2 \neq 0) \text{ and } f(0, 0) = 0$$

is differentiable at the origin.

4. If $u = f(x, y)$ and $x = r \cos \theta$, $y = r \sin \theta$, then if the variables are changed from x, y to r, θ , prove that

$$(i) \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta}\right)^2;$$

$$(ii) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \quad \text{C. H. 1953}$$

[From $x = r \cos \theta$, $y = r \sin \theta$, we get $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$.

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}.$$

$$\begin{aligned} \text{Now } \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \\ &= \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) u, \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \\ &= \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) u. \quad \dots (2) \end{aligned}$$

Squaring (1) and (2) and adding, we get

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta}\right)^2. \quad \dots (i)$$

$$\begin{aligned} \text{Again } \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial u}{\partial x} = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} \\ &\quad + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} \\ &\quad + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \end{aligned}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \frac{\partial u}{\partial y} = \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right)$$

$$= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} \\ + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

Adding

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \quad \dots (ii)]$$

5. Show that the above result (ii) can be put in the form

$$u_{xx} + u_{yy} = \frac{1}{r^2} \left\{ r \frac{\partial}{\partial r} (ru_r) + u_{\theta\theta} \right\}. \quad (C. H. 1962)$$

6. If $V = f(x, y, z)$ and $x = \rho \cos \phi$, $y = \rho \sin \phi$, transform the expression

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

by changing the variables x, y, z to ρ, ϕ, z .

$$[\text{By Ex. 1, } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2},$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

(Note. ρ, ϕ, z are called the cylindric coordinates of a point whose Cartesian coordinates are x, y, z .)

7. If $V = f(x, y, z)$ and $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, show that on changing the variables x, y, z to r, θ, ϕ ,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial V}{\partial \theta} \\ + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}.$$

[In Ex. 3, put $z = r \cos \theta$, $\rho = r \sin \theta$. Then by Ex. 1,

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2};$$

and $\frac{\partial V}{\partial \rho} = \sin \theta \frac{\partial V}{\partial r} + \frac{\cos \theta}{r} \frac{\partial V}{\partial \theta}$, i.e., $\frac{1}{\rho} \frac{\partial V}{\partial \rho} = \frac{1}{r} \frac{\partial V}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial V}{\partial \theta}$.

Put these in the result of Ex. 6.]

8. If $u=f(x, y)$ and the variables x and y are changed to ξ and η by the substitutions $\xi^2=(x-a)^2+y^2$ and $\eta^2=x^2+(y-b)^2$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\xi^2 + \eta^2 - a^2 - b^2}{\xi \eta} \cdot \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{\xi} \frac{\partial u}{\partial \xi} + \frac{1}{\eta} \frac{\partial u}{\partial \eta}.$$

9. If the independent variables are changed from u, v to x, y and then to p, q , prove that

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q} \end{vmatrix}. \quad (C. H. 1953)$$

[We have

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial p}, \quad \frac{\partial u}{\partial q} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial q};$$

$$\frac{\partial v}{\partial p} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial p}, \quad \frac{\partial v}{\partial q} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial q}.$$

$$\begin{aligned} \therefore \begin{vmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q} \end{vmatrix} &= \begin{vmatrix} \frac{\partial u}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial p} & \frac{\partial u}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial q} \\ \frac{\partial v}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial p} & \frac{\partial v}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial q} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{vmatrix} \end{aligned}$$

10. If u is a function of x, y having continuous partial derivatives up to the second order and the variables x, y are changed to ξ, η by the transformation

$$x+y=(\xi+\eta)^n, \quad x-y=(\xi-\eta)^n,$$

prove that

$$(x^2 - y^2) \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right) = \frac{1}{n^2} (\xi^2 - \eta^2) \left(\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} \right). \quad (C. H. 1962)$$

[We have $\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi}$, whence

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial \xi^2} + \frac{\partial x}{\partial \xi} \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \xi} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \xi} \right)$$

$$+ \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial y}{\partial \xi} \left(\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \xi} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \xi} \right). \dots \dots (1)$$

Again from $\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta}$, we get

$$\begin{aligned} \frac{\partial^2 u}{\partial \eta^2} &= \frac{\partial u}{\partial x} \frac{\partial^2 x}{\partial \eta^2} + \frac{\partial x}{\partial y} \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \eta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \eta} \right) \\ &\quad + \frac{\partial u}{\partial y} \frac{\partial^2 y}{\partial \eta^2} + \frac{\partial y}{\partial \eta} \left(\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \eta} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \eta} \right) \dots (2) \end{aligned}$$

$$\begin{aligned} \therefore \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} &= \frac{\partial u}{\partial x} \left(\frac{\partial^2 x}{\partial \xi^2} - \frac{\partial^2 x}{\partial \eta^2} \right) + \frac{\partial u}{\partial y} \left(\frac{\partial^2 y}{\partial \xi^2} - \frac{\partial^2 y}{\partial \eta^2} \right) \\ &\quad + \frac{\partial^2 u}{\partial x^2} \left\{ \left(\frac{\partial x}{\partial \xi} \right)^2 - \left(\frac{\partial x}{\partial \eta} \right)^2 \right\} + 2 \frac{\partial^2 u}{\partial x \partial y} \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} \right) \\ &\quad + \frac{\partial^2 u}{\partial y^2} \left\{ \left(\frac{\partial y}{\partial \xi} \right)^2 - \left(\frac{\partial y}{\partial \eta} \right)^2 \right\}. \dots \dots (3) \end{aligned}$$

Now from $x+y=(\xi+\eta)^n$, and $x-y=(\xi-\eta)^n$, we get

$$x = \frac{1}{2} [(\xi+\eta)^n + (\xi-\eta)^n], \text{ and } y = \frac{1}{2} [(\xi+\eta)^n - (\xi-\eta)^n]$$

$$\therefore \frac{\partial x}{\partial \xi} = \frac{n}{2} [(\xi+\eta)^{n-1} + (\xi-\eta)^{n-1}],$$

$$\frac{\partial^2 x}{\partial \xi^2} = \frac{n(n-1)}{2} [(\xi+\eta)^{n-2} + (\xi-\eta)^{n-2}],$$

$$\frac{\partial x}{\partial \eta} = \frac{n}{2} [(\xi+\eta)^{n-1} - (\xi-\eta)^{n-1}],$$

$$\frac{\partial^2 x}{\partial \eta^2} = \frac{n(n-1)}{2} [(\xi+\eta)^{n-2} - (\xi-\eta)^{n-2}];$$

and from $y = \frac{1}{2} [(\xi+\eta)^n - (\xi-\eta)^n]$, we get

$$\frac{\partial y}{\partial \xi} = \frac{n}{2} [(\xi+\eta)^{n-1} - (\xi-\eta)^{n-1}],$$

$$\frac{\partial^2 y}{\partial \xi^2} = \frac{n(n-1)}{2} [(\xi+\eta)^{n-2} - (\xi-\eta)^{n-2}],$$

$$\frac{\partial y}{\partial \eta} = \frac{n}{2} [(\xi+\eta)^{n-1} + (\xi-\eta)^{n-1}],$$

$$\frac{\partial^2 y}{\partial \eta^2} = \frac{n(n-1)}{2} [(\xi+\eta)^{n-2} + (\xi-\eta)^{n-2}].$$

whence we find

$$\frac{\partial^2 x}{\partial \xi^2} = \frac{\partial^2 x}{\partial \eta^2}, \quad \frac{\partial^2 y}{\partial \xi^2} = \frac{\partial^2 y}{\partial \eta^2}, \quad \frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{\partial^2 y}{\partial \xi \partial \eta}.$$

Also
$$\left(\frac{\partial x}{\partial \xi}\right)^2 - \left(\frac{\partial x}{\partial \eta}\right)^2 = n^2(\xi^2 - \eta^2)^{n-1},$$

$$\left(\frac{\partial y}{\partial \xi}\right)^2 - \left(\frac{\partial y}{\partial \eta}\right)^2 = -n^2(\xi^2 - \eta^2)^{n-1}.$$

Substituting these values in (3), we have

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = n^2(\xi^2 - \eta^2)^{n-1} \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right),$$

or,
$$\frac{1}{n^2}(\xi^2 - \eta^2) \left(\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} \right) = (\xi^2 - \eta^2)^n \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right)$$

$$= (x^2 - y^2) \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right). \quad]$$

11. If $x = c \cosh \xi \cos \eta$, $y = c \sinh \xi \sin \eta$, show that

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{2} c^2 (\cosh 2\xi - \cos 2\eta) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

(C. H. 1965)

[Adding relations (1) and (2) of Ex. 7, we get

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} &= \frac{\partial u}{\partial x} \left(\frac{\partial^2 x}{\partial \xi^2} + \frac{\partial^2 x}{\partial \eta^2} \right) + \frac{\partial u}{\partial y} \left(\frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \eta^2} \right) + \frac{\partial^2 u}{\partial x^2} \left(\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial x}{\partial \eta} \right)^2 \right) \\ &\quad + 2 \frac{\partial^2 u}{\partial x \partial y} \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} + \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} \right) + \frac{\partial^2 u}{\partial y^2} \left\{ \left(\frac{\partial y}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 \right\} \dots \quad (1) \end{aligned}$$

From $x = c \cosh \xi \cos \eta$

$$\frac{\partial x}{\partial \xi} = c \sinh \xi \cos \eta,$$

$$\frac{\partial^2 x}{\partial \xi^2} = c \cosh \xi \cos \eta,$$

$$\frac{\partial x}{\partial \eta} = -c \cosh \xi \sin \eta,$$

$$\frac{\partial^2 x}{\partial \eta^2} = -c \cosh \xi \cos \eta,$$

And from $y = c \sinh \xi \sin \eta$

$$\frac{\partial y}{\partial \xi} = c \cosh \xi \sin \eta$$

$$\frac{\partial^2 y}{\partial \xi^2} = c \sinh \xi \sin \eta$$

$$\frac{\partial y}{\partial \eta} = c \sinh \xi \cos \eta$$

$$\frac{\partial^2 y}{\partial \eta^2} = -c \sinh \xi \sin \eta$$

Hence
$$\frac{\partial^2 x}{\partial \xi^2} + \frac{\partial^2 x}{\partial \eta^2} = 0, \quad \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \eta^2} = 0, \quad \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} + \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} = 0.$$

$$\begin{aligned}\text{Also } \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta}\right)^2 &= c^2 \sinh^2 \xi \cos^2 \eta + c^2 \cosh^2 \xi \sin^2 \eta \\ &= c^2 (\cosh^2 \xi - 1) \cos^2 \eta + c^2 \cosh^2 \xi (1 - \cos^2 \eta) \\ &= c^2 (\cosh^2 \xi - \cos^2 \eta) = \frac{c^2}{2} (\cosh 2\xi - \cos 2\eta),\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 &= c^2 \cosh^2 \xi \sin^2 \eta + c^2 \sinh^2 \xi \cos^2 \eta \\ &= \frac{c^2}{2} (\cosh 2\xi - \cos 2\eta).\end{aligned}$$

Substituting these values in (1), we get

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{2} c^2 (\cosh 2\xi - \cos 2\eta) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad]$$

12. A function $f(x, y)$ when expressed in terms of the new variables u, v defined by the equations $x = \frac{1}{2}(u+v)$, $y^2 = uv$ becomes $g(u, v)$; prove that

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{4} \left\{ \frac{\partial^2 f}{\partial x^2} + 2 \frac{x}{y} \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y} \frac{\partial f}{\partial y} \right\}. \quad (C. H. 1965)$$

$$[\text{We have } \frac{\partial x}{\partial u} = \frac{1}{2}, \frac{\partial x}{\partial v} = \frac{1}{2}, \frac{\partial y}{\partial u} = \frac{1}{2} \sqrt{\frac{v}{u}}, \frac{\partial y}{\partial v} = \frac{1}{2} \sqrt{\frac{u}{v}}.]$$

$$\text{Now } \frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{1}{2} \sqrt{\frac{v}{u}} \frac{\partial f}{\partial y}.$$

$$\therefore \frac{\partial^2 g}{\partial u \partial v} = \frac{\partial}{\partial v} \left[\frac{1}{2} \frac{\partial f}{\partial x} + \frac{1}{2} \sqrt{\frac{v}{u}} \frac{\partial f}{\partial y} \right]$$

$$\begin{aligned}&= \frac{1}{2} \left\{ \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial y}{\partial v} \right\} + \frac{1}{4} \frac{1}{\sqrt{uv}} \frac{\partial f}{\partial y} \\ &\quad + \frac{1}{2} \sqrt{\frac{v}{u}} \left\{ \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial v} \right\}\end{aligned}$$

$$\begin{aligned}&= \frac{1}{4} \frac{\partial^2 f}{\partial x^2} + \frac{1}{4} \sqrt{\frac{u}{v}} \frac{\partial^2 f}{\partial x \partial y} + \frac{1}{4} \sqrt{\frac{v}{u}} \frac{\partial^2 f}{\partial x \partial y} \\ &\quad + \frac{1}{4} \sqrt{\frac{u}{v}} \sqrt{\frac{v}{u}} \frac{\partial^2 f}{\partial y^2} + \frac{1}{4y} \frac{\partial f}{\partial y}\end{aligned}$$

$$= \frac{1}{4} \left\{ \frac{\partial^2 f}{\partial x^2} + 2 \frac{x}{y} \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y} \frac{\partial f}{\partial y} \right\}. \quad]$$

13. If u be a homogeneous function in x, y, z of degree n , prove that $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ are each a homogeneous function in x, y, z of degree $n-1$.

14. If u is a homogeneous function of the n th degree in x, y, z and if $u = f(\xi, \eta, \zeta)$ where ξ, η, ζ are the partial derivatives of u with regard to x, y, z respectively. prove that

$$\xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \zeta \frac{\partial u}{\partial \zeta} = \frac{n}{n-1} u.$$

[Since $\xi = \frac{\partial u}{\partial x}$, $\eta = \frac{\partial u}{\partial y}$, $\zeta = \frac{\partial u}{\partial z}$, ξ, η, ζ are homogeneous functions of degree $n-1$.

By Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu, \text{ i.e., } x\xi + y\eta + z\zeta = nu.$$

Since each of x, y, z is a function of ξ, η, ζ ,

$$\xi = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x},$$

$$\eta = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y},$$

$$\zeta = \frac{\partial u}{\partial z} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial z} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} + \frac{\partial u}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial z}.$$

$$\therefore nu = x\xi + y\eta + z\zeta$$

$$\begin{aligned} &= \frac{\partial u}{\partial \xi} \left(x \frac{\partial \xi}{\partial x} + y \frac{\partial \xi}{\partial y} + z \frac{\partial \xi}{\partial z} \right) + \frac{\partial u}{\partial \eta} \left(x \frac{\partial \eta}{\partial x} + y \frac{\partial \eta}{\partial y} + z \frac{\partial \eta}{\partial z} \right) \\ &\quad + \frac{\partial u}{\partial \zeta} \left(x \frac{\partial \zeta}{\partial x} + y \frac{\partial \zeta}{\partial y} + z \frac{\partial \zeta}{\partial z} \right) \\ &= (n-1) \left(\xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \zeta \frac{\partial u}{\partial \zeta} \right). \end{aligned}$$

$$\text{Hence } \xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \zeta \frac{\partial u}{\partial \zeta} = \frac{n}{n-1} u.]$$

15. If $x^2 = vw$, $y^2 = wu$, $z^2 = uv$ and $f(x, y, z) = \phi(u, v, w)$, then
 $xf_x + yf_y + zf_z = u\phi_u + v\phi_v + w\phi_w.$ (Math. Tripos 1933)

[From the given relations we get

$$u = \pm \frac{yz}{x}, v = \pm \frac{zx}{y}, w = \pm \frac{xy}{z};$$

so that each of u, v, w is a homogeneous function of degree 1.

$$\therefore xu_x + yu_y + zu_z = u, xv_x + yv_y + zv_z = v, xw_x + yw_y + zw_z = w.$$

$$\text{Now } f_x = \phi_u u_x + \phi_v v_x + \phi_w w_x, f_y = \phi_u u_y + \phi_v v_y + \phi_w w_y, \\ f_z = \phi_u u_z + \phi_v v_z + \phi_w w_z.$$

$$\therefore xf_x + yf_y + zf_z = \phi_u (xu_x + yu_y + zu_z) + \phi_v (xv_x + yv_y + zv_z) \\ + \phi_w (xw_x + yw_y + zw_z) \\ = u\phi_u + v\phi_v + w\phi_w.]$$

$$16. \text{ If } u = \operatorname{cosec}^{-1} \sqrt{\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}}, \text{ then}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12} \left(\frac{13}{12} + \frac{\tan^2 u}{12} \right).$$

$$\left[\text{We have } \operatorname{cosec}^2 u = \frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}, \text{ a homogeneous function of degree } \frac{1}{6}; \right.$$

$$\therefore x \frac{\partial}{\partial x} (\operatorname{cosec}^2 u) + y \frac{\partial}{\partial y} (\operatorname{cosec}^2 u) = \frac{1}{6} \operatorname{cosec}^2 u,$$

$$\text{or } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{12} \tan u.$$

$$\therefore \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = -\frac{1}{12} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \tan u,$$

$$\text{or, } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \\ = -\frac{\sec^2 u}{12} \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right),$$

$$\text{or, } x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = - \left(1 + \frac{\sec^2 u}{12} \right) \left(-\frac{1}{12} \tan u \right) \\ = \frac{\tan u}{12} \left(1 + \frac{1 + \tan^2 u}{12} \right) = \frac{\tan u}{12} \left(\frac{13}{12} + \frac{\tan^2 u}{12} \right).]$$

17. If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4 \sin^2 u) \sin 2u.$$

18. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, then

$$\frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = r^2 \sin \theta,$$

where
$$\frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}.$$

19. If $x = \lambda + \mu + v$, $y = \lambda^2 + \mu^2 + v^2$, $z = \lambda^3 + \mu^3 + v^3$, prove that

$$\frac{\partial \lambda}{\partial x} = \frac{\mu v (\mu - v)}{(\lambda - \mu)(\mu - v)(v - \lambda)};$$

also find $\frac{\partial \lambda}{\partial x}$, $\frac{\partial \lambda}{\partial z}$.

[Differentiate the three equations with respect to x ;

eliminate $\frac{\partial \mu}{\partial x}$, $\frac{\partial v}{\partial x}$.]

20. If $P dx + Q dy + R dz$ can be made a perfect differential of some function of x, y, z by multiplying each term by a common factor, show that

$$P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = 0. \quad (C. H. 1954)$$

[See Murray, Differential Equations, Art. 102.]

21. Prove the converse of Euler's theorem on homogeneous functions, that is, given $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$, prove that u is a homogeneous function of degree n in x, y, z . (C. H. 1960, 1962)

[Put $\xi = \frac{x}{z}$, $\eta = \frac{y}{z}$, $\zeta = z$, the equation $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$ transforms into $\zeta \frac{\partial v}{\partial \zeta} = nv$, where $u(x, y, z) = v(\xi, \eta, \zeta)$. We have

$$x \frac{\partial u}{\partial x} = x \left\{ \frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial x} \right\} = \xi \frac{\partial v}{\partial \xi},$$

$$y \frac{\partial u}{\partial y} = y \left\{ \frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial y} \right\} = \eta \frac{\partial v}{\partial \eta},$$

$$z \frac{\partial u}{\partial z} = z \left\{ \frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right\} = -\xi \frac{\partial v}{\partial \xi} - \eta \frac{\partial v}{\partial \eta} + \zeta \frac{\partial v}{\partial \zeta}.$$

Thus $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$ transforms into $\zeta \frac{\partial v}{\partial \zeta} = nv$.

$\therefore v = \zeta^n c$ where c is constant with respect to ζ but may involve ξ, η . Put $c = \phi(\xi, \eta)$, we get $v = \zeta^n \phi(\xi, \eta) = z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right)$ showing that v and therefore u is a homogeneous function of degree n in x, y, z .

This is an alternative treatment of the theorem of Art. 17'14.]

MISCELLANEOUS THEOREMS

18'1. Implicit Functions.

Given a relation of the form $f(x, y)=0$ it is not always possible to obtain from this a relation of the form $y=\phi(x)$, as the following examples will show.

Ex. 1. $e^{x-y}=0$ can never be solved for y in terms of x .

Ex. 2. $x^2+y^2+1=0$ does not give a value of y corresponding to any value of x .

Ex. 3. From $x^2+y^2=0$, y can be determined only for $x=0$ and for no other values of x .

The existence of an explicit function $y=f(x)$ corresponding to the implicit relation $\phi(x, y)=0$ is given by the following theorem.

Theorem : If $\phi(x, y)$ be continuous together with its first partial derivatives in the neighbourhood of a point (x_0, y_0) where $\phi(x_0, y_0)=0$ and $\phi_y(x_0, y_0) \neq 0$, then there exists a function $y=f(x)$ continuous in the neighbourhood of x_0 , satisfying $y_0=f(x_0)$ such that $\phi\{x, f(x)\}=0$.

[We state the theorem without proof. For an interesting discussion of implicit functions, the student is referred to Hardy, Pure Mathematics, Art. 109.]

18'2. Mean Value Theorem for a function of two variables.

Let $f(x, y)$ be a differentiable function of the two variables x and y defined in a domain containing the points (a, b) , $(a+h, b+k)$, such that the line joining them lies wholly in the domain. Then

$$f(a+h, b+k)-f(a, b)=hf_x(a+\theta h, b+\theta k)+kf_y(a+\theta h, b+\theta k),$$

where $0 < \theta < 1$.

Let us write $x=a+ht$, $y=b+kt$, so that $f(x, y)=f(a+ht, b+kt)=\phi(t)$, a function of t . Applying the mean value theorem to the function $\phi(t)$ of the single variable t in $(0, t)$, we get

$$\phi(t)-\phi(0)=t\phi'(\theta t), \quad (0 < \theta < 1).$$

Now by the theorem of the total differential coefficient

(Art. 17'12)

$$\begin{aligned}\phi'(t) &= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \\ &= hf_x(a+ht, b+kt) + kf_y(a+ht, b+kt).\end{aligned}$$

$$\therefore \phi(t) - \phi(0) = t\{hf_x(a + \theta ht, b + \theta kt) + kf_y(a + \theta ht, b + \theta kt)\}$$

On putting $t=1$, we get the mean value theorem, viz.,

$$f(a+h, b+k) - f(a, b) = hf_x(a + \theta h, b + \theta k) + kf_y(a + \theta h, b + \theta k).$$

Cor. If the partial derivatives f_x, f_y of a function exist and have the value 0 at every point of the domain, then $f(x, y)$ is a constant.

Note. The theorem admits of easy generalisation to functions of any number of variables.

18.3. A Lemma.

Let $x = a + ht$, $y = b + kt$, then $f(x, y) = f(a + ht, b + kt)$ is a composite function of the single variable t . By Art. 17.12.

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}. \\ \frac{d^2 f}{dt^2} &= \frac{d}{dt} \left(\frac{df}{dt} \right) = h \frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) + k \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) \\ &= h \left\{ \frac{\partial^2 f}{\partial x^2} \frac{dx}{dt} + \frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dt} \right\} + k \left\{ \frac{\partial^2 f}{\partial x \partial y} \frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2} \frac{dy}{dt} \right\} \\ &= h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x, y). \end{aligned}$$

It follows by induction that

$$\frac{d^r f}{dt^r} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^r f(x, y).$$

18.4. Taylor's theorem for a function of two variables.

Let $f(x, y)$ be a differentiable function of the two variables x and y having differentials up to the n th order for all values of x between a and $a+h$ and for all values of y between b and $b+k$. Let t be a new independent variable and let us put

$$x = a + ht, \quad y = b + kt.$$

$f(x, y)$ is then a composite function of t , say $\phi(t)$, which has derivatives up to the n th order and these derivatives are continuous up to the $(n-1)$ th order inclusive in the interval $(0, 1)$. Applying the Maclaurin formula to the function $\phi(t)$ of the single variable t with Lagrange's form of the remainder, we get

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(0) + \dots + \frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0) \\ + \frac{t^n}{n!}\phi^{(n)}(\theta t), \quad 0 < \theta < 1.$$

On putting $t=1$, we have

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \dots + \frac{1}{(n-1)!}\phi^{(n-1)}(0) + \frac{1}{n!}\phi^{(n)}(\theta) \quad \dots \quad (1)$$

By the above lemma

$$\frac{d^r f}{dt^r} = \phi^{(r)}(t) = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^r f(x, y).$$

Putting $t=0$, we have $x=a$, $y=b$ and we get

$$\phi^{(r)}(0) = \left(h \frac{\partial}{\partial a} + k \frac{\partial}{\partial b} \right)^r f(a, b).$$

For $t=\theta$, $x=a+\theta h$, $y=b+\theta k$ and so

$$\phi^{(r)}(\theta) = \left(h \frac{\partial}{\partial a} + k \frac{\partial}{\partial b} \right)^r f(a+\theta h, b+\theta k).$$

Putting these values in (1), we have the formula of Taylor, viz.,

$$f(a+h, b+k) = f(a, b) + \left(h \frac{\partial}{\partial a} + k \frac{\partial}{\partial b} \right) f(a, b) \\ + \frac{1}{2!} \left(h \frac{\partial}{\partial a} + k \frac{\partial}{\partial b} \right)^2 f(a, b) + \dots \\ + \frac{1}{(n-1)!} \left(h \frac{\partial}{\partial a} + k \frac{\partial}{\partial b} \right)^{n-1} f(a, b) \\ + \frac{1}{n!} \left(h \frac{\partial}{\partial a} + k \frac{\partial}{\partial b} \right)^n f(a+\theta h, b+\theta k), \quad \dots \quad (2)$$

where $0 < \theta < 1$.

Replacing a and b by x and y , the formula may be written in the form

$$f(x+h, y+k) = f(x, y) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x, y) \\ + \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x, y) + \dots \\ + \frac{1}{(n-1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n-1} f(x, y) \\ + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x+\theta h, y+\theta k), \quad \dots \quad (3)$$

where $0 < \theta < 1$.

The formula assumes the differentiability and therefore the continuity of the derivatives up to the $(n-1)$ th order, but not the continuity of the derivative of the n th order.

The formula can be easily extended to functions of any number of variables. Thus for a function of three variables

$$\begin{aligned} f(x+h, y+k, z+l) &= f(x, y, z) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z} \right) f(x, y, z) \\ &+ \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z} \right)^2 f(x, y, z) + \dots \\ &+ \frac{1}{(n-1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z} \right)^{n-1} f(x, y, z) \\ &+ \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} + l \frac{\partial}{\partial z} \right)^n f(x+\theta h, y+\theta k, z+\theta l), \end{aligned}$$

$(0 < \theta < 1)$.

18'5. Maclaurin formula for a function of two variables.

Putting $a=b=0$, $h=x$ and $k=y$ in formula (2) of the preceding article, we get

$$\begin{aligned} f(x, y) &= f(0, 0) + \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) f_{0,0} + \frac{1}{2!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 f_{0,0} + \dots \\ &+ \frac{1}{(n-1)!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^{n-1} f_{0,0} \\ &+ \frac{1}{n!} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^n f_{\theta x, \theta y}, \quad (0 < \theta < 1). \end{aligned}$$

The sub-scripts of f in the different terms of the above mean that after calculating the partial derivatives of $f(x, y)$, we have to put for x and y in them values indicated by the sub-scripts.

The formula admits of easy extension to functions of more than two variables.

18'6. Extrema of a function of two independent variables.

The function $f(x, y)$ of the two independent variables x and y is a *maximum* at the point (x, y) if

$$f(x+h, y+k) - f(x, y) < 0 \text{ for } |h| < \delta, |k| < \delta;$$

and a *minimum* if

$$f(x+h, y+k) - f(x, y) > 0 \text{ for } |h| < \delta, |k| < \delta.$$

Since x and y can be made to vary independently, it is clear that if $f(x, y)$ has an extremum at (x, y) when both x and y vary, it must have an extremum at the same point when only one of x and y is made to vary; we must therefore have for a maximum at (x, y)

$$\begin{aligned} f(x+h, y) - f(x, y) &< 0, \quad |h| < \delta, \\ f(x, y+k) - f(x, y) &< 0, \quad |k| < \delta; \end{aligned}$$

and for a minimum at the point

$$\begin{aligned} f(x+h, y) - f(x, y) &> 0, \quad |h| < \delta, \\ f(x, y+k) - f(x, y) &> 0, \quad |k| < \delta; \end{aligned}$$

in other words, for an extremum at (x, y) , we must have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

Now just as a function $f(x)$ can have an extremum at a point where $f'(x)$ does not exist (See Ex. 1, p. 133), so a function $f(x, y)$ of two independent variables x and y may have an extremum at a point where f_x and f_y do not exist. For example the function $f(x, y) = |x| + |y|$ has a minimum at $(0, 0)$, although f_x and f_y do not exist at the point. We thus have the following *necessary* condition for an extremum of $f(x, y)$.

The only points where the function $f(x, y)$ can have an extremum are those where each of the partial derivatives f_x and f_y vanish or cease to exist.

Let us next suppose that f_x and f_y exist at the point (x, y) . On solving the simultaneous equations $f_x = 0, f_y = 0$, we generally find a number of solutions. Let $x = a, y = b$ be one of them. It now remains to see if $f(x, y)$ is really a maximum or a minimum or merely stationary at the point (a, b) .

For an extremum at (a, b) , $f(a+h, b+k) - f(a, b)$ must maintain a constant sign for sufficiently small values of $|h|$ and $|k|$.

Supposing the first partial derivatives to be differentiable at the point (a, b) and consequently existent in a neighbourhood of the point, we have by the mean value theorem (Art. 18'2)

$$f(a+h, b+k) - f(a, b) = hf_x(a+\theta h, b+\theta k) + kf_y(a+\theta h, b+\theta k).$$

We put

$$\rho = \sqrt{h^2 + k^2}, \quad h = \rho \sin \alpha, \quad k = \rho \cos \alpha,$$

so that ρ is an indefinitely small positive quantity tending to zero with h and k and α an arbitrary angle depending on h and k which are arbitrary.

We further write

$$r = \frac{\partial^2 f(a, b)}{\partial a^2}, \quad s = \frac{\partial^2 f(a, b)}{\partial a \partial b}, \quad t = \frac{\partial^2 f(a, b)}{\partial b^2}.$$

The first partial derivatives being supposed differentiable, we have (Art. 17'8)

$$f_x(a + \theta h, b + \theta k) - f_x(a, b) = f_{xx}(a + \theta h, b + \theta k) \theta h + f_{xy}(a + \theta h, b + \theta k) \theta k + \epsilon_1 \rho,$$

and similarly

$$f_y(a + \theta h, b + \theta k) = \theta (sh + tk + \epsilon_2 \rho),$$

where ϵ_1 and ϵ_2 tend to zero with ρ . Substituting these values, we get

$$\begin{aligned} f(a + h, b + k) - f(a, b) &= \theta (rh^2 + 2shk + tk^2 + \epsilon_1 \rho h + \epsilon_2 \rho k) \\ &= \theta \rho^2 (r \sin^2 \alpha + 2s \sin \alpha \cos \alpha + t \cos^2 \alpha + \epsilon) \quad \dots (1) \end{aligned}$$

where $\epsilon = \epsilon_1 \sin \alpha + \epsilon_2 \cos \alpha$ so that $\epsilon \rightarrow 0$ as $\epsilon_1, \epsilon_2 \rightarrow 0$, i.e., as $\rho \rightarrow 0$.

Now θ being positive, we have the following cases to consider :

(a) Suppose the expression $r \sin^2 \alpha + 2s \sin \alpha \cos \alpha + t \cos^2 \alpha$ does not vanish. As it is a continuous function of α , it will have a definite sign and its absolute value shall be greater than a positive number m . So the sign of the expression will be the same as that of (1) from a sufficiently small value of ρ since we can make $|\epsilon| < m$. In this case therefore there will be a maximum or a minimum according as the sign of the expression is negative or positive.

(b) If the expression changes sign, as the sign of (1) is still the same as that of this expression from a sufficiently small value of ρ , there will be neither a maximum nor a minimum.

(c) Lastly if the expression without changing its sign vanish for certain values of α , then the sign of (1) for such values of α depends on that of ϵ , which being unknown, we cannot conclude anything.

From the above discussions it is clear that there is surely an extremum of $f(x, y)$ when $r \sin^2 \alpha + 2s \sin \alpha \cos \alpha + t \cos^2 \alpha$ does not vanish.

$$\text{Now } r \sin^2 \alpha + 2s \sin \alpha \cos \alpha + t \cos^2 \alpha = \frac{r h^2 + 2s h k + t k^2}{\rho^2}.$$

Omitting a positive multiplier,

$$r h^2 + 2s h k + t k^2 = \frac{(hr + ks)^2 + k^2(rt - s^2)}{r}.$$

Since $(hr + ks)^2$ is positive, the numerator would be positive if $rt > s^2$ and then $f(a+h, b+k) - f(a, b)$ would be positive if r is positive and negative if r is negative. Therefore if $rt > s^2$ and r is negative, $f(a, b)$ is maximum at (a, b) while if $rt > s^2$ and r is positive, $f(a, b)$ is minimum at (a, b) .

By rewriting the expression with t in the denominator, it can be seen that if $rt > s^2$ and t is negative, we get a maximum, and if t is positive, we get a minimum.

If $rt = s^2$, the case is a doubtful one; for $(hr + ks)^2$ vanishes if $\frac{h}{k} = -\frac{s}{r}$ and in this case $f(a, b)$ may not be an extremum at (a, b) .

If $rt < s^2$, the expression has different signs under the two hypotheses $k=0$ and $hr + ks=0$, and cannot therefore maintain the same sign for all values of h and k such that $|h| < \delta$, $|k| < \delta$. Hence there is no extremum at (a, b) .

A set of *sufficient* conditions for an extremum at (x, y) is:

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2.$$

The necessity of the last condition was first established by Lagrange and is known as *Lagrange's condition*.

Note. Lagrange's condition, viz., $f_{xx} \cdot f_{yy} < (f_{xy})^2$ is a *sufficient* condition but not a *necessary* one, $f(x, y)$ may have an extremum at a point where $f_{xx} \cdot f_{yy} = (f_{xy})^2$. See Ex. 5(i) below.

187. Extrema of a function of several variables under subsidiary conditions. Lagrange's method of undetermined multipliers.

Let $u = f(x, y, z, w)$,

$$\text{where } \begin{cases} \phi(x, y, z, w) = 0, \\ \psi(x, y, z, w) = 0. \end{cases} \quad \dots \quad \dots \quad (1)$$

For an extremum $du = 0$; hence

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial w} dw = 0. \quad \dots \quad (2)$$

Also
$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz + \frac{\partial \phi}{\partial w} dw = 0, \quad \dots \quad (3)$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz + \frac{\partial \psi}{\partial w} dw = 0. \quad \dots \quad (4)$$

Multiplying (3) and (4) by the arbitrary quantities λ and μ and adding (2), (3), (4), we get

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} + \mu \frac{\partial \psi}{\partial x} \right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} + \mu \frac{\partial \psi}{\partial y} \right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} + \mu \frac{\partial \psi}{\partial z} \right) dz + \left(\frac{\partial f}{\partial w} + \lambda \frac{\partial \phi}{\partial w} + \mu \frac{\partial \psi}{\partial w} \right) dw = 0. \quad (5)$$

As λ and μ are arbitrary, we may suppose them so determined as to make

$$\left. \begin{aligned} \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} + \mu \frac{\partial \psi}{\partial z} &= 0, \\ \frac{\partial f}{\partial w} + \lambda \frac{\partial \phi}{\partial w} + \mu \frac{\partial \psi}{\partial w} &= 0. \end{aligned} \right\} \quad \dots \quad (6)$$

We thus get

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} + \mu \frac{\partial \psi}{\partial x} \right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} + \mu \frac{\partial \psi}{\partial y} \right) dy = 0. \quad \dots \quad (7)$$

Now since the four variables x, y, z, w are connected by the two equations (1), we can find two of them in terms of the other two; we can thus regard u as function of two independent variables, say x and y . As these can be made to vary independently of each other, it follows that for an extremum, the coefficients of dx and dy in (7) must vanish separately. We thus have

$$\left. \begin{aligned} \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} + \mu \frac{\partial \psi}{\partial x} &= 0, \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} + \mu \frac{\partial \psi}{\partial y} &= 0. \end{aligned} \right\} \quad \dots \quad (8)$$

From (6),

$$\frac{\lambda}{\psi_z f_w - \psi_w f_z} = \frac{\mu}{f_z \phi_w - f_w \phi_z} = \frac{1}{\phi_z \psi_w - \phi_w \psi_z}.$$

Hence in order that λ and μ may be determinate

$$\phi_z \psi_w - \phi_w \psi_z \neq 0,$$

$$\text{or } \begin{vmatrix} \phi_z & \psi_z \\ \phi_w & \psi_w \end{vmatrix} \neq 0,$$

$$\text{or, } \frac{\partial(\phi, \psi)}{\partial(z, w)} \neq 0.$$

The equations (1), (6), and (8) are theoretically sufficient to determine x, y, z, w, λ, μ , and thus to furnish a solution of the problem in general.

Hence we have the following theorem :

If $f(x, y, z, w)$ has an extreme value at (ξ, η, ζ, ρ) subject to the subsidiary conditions

$$\phi(x, y, z, w) = 0, \quad \psi(x, y, z, w) = 0,$$

and if at that point

$$\frac{\partial(\phi, \psi)}{\partial(z, w)} \neq 0,$$

then two numbers λ and μ exist such that at the point (ξ, η, ζ, ρ) , the equations

$$\begin{aligned} f_x + \lambda \phi_x + \mu \psi_x &= 0, \\ f_y + \lambda \phi_y + \mu \psi_y &= 0, \\ f_z + \lambda \phi_z + \mu \psi_z &= 0, \\ f_w + \lambda \phi_w + \mu \psi_w &= 0, \end{aligned}$$

and also the subsidiary conditions are satisfied.

Note 1. If we put $F = f + \lambda \phi + \mu \psi$ where x, y, z, w are considered as independent variables and λ, μ as constants, the equations (8) and (6) are easily obtained as $F_x = 0, F_y = 0, F_z = 0, F_w = 0$.

F is sometimes called the Lagrangian function.

Note 2. The consideration of the function F is also convenient for the discussion of the sign of d^2f . In fact we have $f = F$ by virtue of the equations (1), whence we get $d^2f = d^2F$. Replacing d^2F by its expression

$$\left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz + \frac{\partial}{\partial w} dw \right)^2 F + \frac{\partial F}{\partial z} d^2z + \frac{\partial F}{\partial w} d^2w$$

where the terms involving d^2z, d^2w disappear by virtue of the equations (6), what remains is

$$d^2f = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz + \frac{\partial}{\partial w} dw \right)^2 F.$$

We can thus replace d^2f by d^2F and calculate d^2F as if the variables x, y, z, w were independent and λ, μ constants. It is to be understood that for the discussion of the sign of d^2f , it may perhaps be necessary to eliminate the first differentials dz, dw of the dependent variables; but the introduction of F has the advantage of eliminating immediately their second differentials.

Examples A.

1. If the partial derivatives f_x, f_y of a function $f(x, y)$ are continuous in R , prove that

$$f(x+h, y+k) - f(x, y) = hf_x(\xi, \eta) + kf_y(\xi, \eta)$$

where (x, y) and $(x+h, y+k)$ and the line-segment joining them lie in R and (ξ, η) is an intermediate point on the line-segment.

If the partial derivatives f_x, f_y of a function exist and have the value 0 at every point of R , prove that $f(x, y)$ is a constant.

(C. H. 1961)

2. Establish Taylor's theorem in finite form for a function $f(x, y)$ of two variables, stating the condition imposed upon $f(x, y)$.

(C. H. 1963)

3. Apply the Maclaurin formula to expand $f(x, y)$ in a finite series of four terms when

$$(i) \quad f(x, y) = e^{ax} \cos by,$$

$$(ii) \quad f(x, y) = e^{ax} \sin by.$$

(C. H. 1964)

$$\left[(i) \quad 1 + ax + \frac{1}{2!}(a^2x^2 - b^2y^2) + \frac{e^{a\theta x}}{3!}\{(a^3x^3 - 3ab^2xy^2) \cos b\theta y - (3a^2bx^2y - b^3y^3) \sin b\theta y\}, \right.$$

$$(ii) \quad by + abxy + \frac{e^{a\theta y}}{3!}\{(3a^2bx^2y - b^3y^3) \cos b\theta y + (a^3x^3 - 3ab^2xy^2) \sin b\theta y\}$$

4. (i) Find the minimum value of $x^2 + y^2 + (x+y+1)^2$.

(C. H. 1966)

(ii) Find the maximum or minimum value of

$$f(x, y) = x^2 + xy + y^2 + ax + by,$$

and determine whether the value you get is maximum or minimum.

(C. H. 1964)

$$[(i) \text{ minimum value} = \frac{1}{3} \text{ at } \left(-\frac{1}{3}, -\frac{1}{3}\right).$$

$$(ii) \text{ minimum at } \left(\frac{b-2a}{3}, \frac{a-2b}{3}\right)].$$

5. (i) Show that $f(x, y) = y^2 + 2x^2y + x^3$ has a minimum at $(0, 0)$ where $f_{xx}f_{yy} - (f_{xy})^2 = 0$.

(ii) The equation to a surface is $z = ay^2 - x^3$; show that the ordinate z of the surface is stationary at a point but is neither a maximum nor a minimum there. (C. H. 1965)

6. If two variables x and y are connected by the relation $ax^2 + by^2 = ab$, show that the maximum and minimum values of the function $x^2 + xy + y^2$ will be the values of u given by the equation $4(u-a)(u-b) = ab$.

$$[\text{Here } F = x^2 + xy + y^2 + \lambda(ax^2 + by^2 - ab).$$

$$F_x = 2x + y + 2a\lambda x = 0, F_y = x + 2y + 2b\lambda y = 0.$$

$$\therefore -\lambda = \frac{2x+y}{2ax} = \frac{x+2y}{2by} = \frac{x^2+xy+y^2}{ab} = \frac{u}{ab},$$

where u is an extreme value of $x^2 + xy + y^2$.

$$\text{From } F_x = 0, 2x \left(1 - \frac{u}{b}\right) = -y; \text{ from } F_y = 0, 2y \left(1 - \frac{u}{a}\right) = -x,$$

whence $4(u-a)(u-b) = ab$.]

7. In a triangle the area Δ and the semi-perimeter s are fixed. Show that any maximum or minimum of one of the sides is a root of the equation $s(x-s)x^2 + 4\Delta^2 = 0$. (C. H. 1966)

[Let x, y, z be the sides of the triangle. We have $x+y+z=2s$ and $s(s-x)(s-y)(s-z) = \Delta^2$, i.e., $\log(s-x) + \log(s-y) + \log(s-z) = 2 \log \Delta - \log s$.

Any side x may be taken as the independent variable; the other two sides y and z are then dependent upon it by virtue of the above two relations. Now these two relations give on differentiating, $dx + dy$

$+dz=0$ and $\frac{dx}{s-x} + \frac{dy}{s-y} + \frac{dz}{s-z} = 0$. For an extremum of x , $dx=0$

and so $dy+dz=0$, $\frac{dy}{s-y} + \frac{dz}{s-z} = 0$. Eliminating dy, dz , we get

$\frac{1}{s-y} = \frac{1}{s-z} = \frac{2}{x}$, or $s-y = \frac{x}{2}$, $s-z = \frac{x}{2}$. Putting these values in $s(s-x)(s-y)(s-z) = \Delta^2$, we get $s(x-s)x^2 + 4\Delta^2 = 0$.]

8. Show that the maximum and minimum values of

$\frac{ax^2 + 2bxy + cy^2}{ex^2 + 2fxy + gy^2}$ ($eg + f^2 > 0$) are the roots of the equation in u :

$$u^2(eg - f^2) - u(ag - 2bf + ec) + ac - b^2 = 0. \quad (C. H. 1962)$$

[Find the extreme values of $ax^2 + 2bxy + cy^2$ subject to the condition $ex^2 + 2fxy + gy^2 = 1$.]

9. Show that the maximum and minimum values of r^2 where $r^2 = a^2x^2 + b^2y^2 + c^2z^2$, $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$ are given by

$$\frac{l^2}{a^2 - r^2} + \frac{m^2}{b^2 - r^2} + \frac{n^2}{c^2 - r^2} = 0.$$

$$[F = a^2x^2 + b^2y^2 + c^2z^2 + \lambda(x^2 + y^2 + z^2 - 1) + \mu(lx + my + nz).$$

$$\therefore F_x = 2a^2x + 2\lambda x + \mu l = 0, \quad F_y = 2b^2y + 2\lambda y + \mu m,$$

$$F_z = 2c^2z + 2\lambda z + \mu n = 0.$$

Multiplying these by x, y, z respectively and adding, we get $\lambda = -r^2$ where r^2 is an extreme value of $a^2x^2 + b^2y^2 + c^2z^2$. Putting this value of λ in $F_x = F_y = F_z = 0$, we have $\frac{2x}{\mu} = -\frac{l}{a^2 - r^2}$, $\frac{2y}{\mu} = -\frac{m}{b^2 - r^2}$,

$$\frac{2z}{\mu} = -\frac{n}{c^2 - r^2}.$$

Multiplying respectively by l, m, n and adding, we have

$$\left[\frac{l^2}{a^2 - r^2} + \frac{m^2}{b^2 - r^2} + \frac{n^2}{c^2 - r^2} = 0. \right]$$

10. Find the minimum value of $x^2 + y^2 + z^2$ subject to conditions $ax + by + cz = 1$, $a'x + b'y + c'z = 1$. (C.H. 1956)

[The minimum value is a root of

$$\begin{bmatrix} u & 1 & 1 & = 0. \\ 1 & \Sigma a^2 & \Sigma aa' & \\ 1 & \Sigma aa' & \Sigma a'^2 & \end{bmatrix}$$

11. Divide the positive number a into three parts x, y, z such that $f = x^m y^n z^p$ may be a maximum, (m, n, p being positive).

$$\left[\text{We have } \frac{df}{f} = \frac{m}{x} dx + \frac{n}{y} dy + \frac{p}{z} dz = 0, \right.$$

$$\text{and from } x + y + z = a, \quad dx + dy + dz = 0.$$

$$\therefore \left(\lambda \frac{m}{x} + 1 \right) dx + \left(\lambda \frac{n}{y} + 1 \right) dy + \left(\lambda \frac{p}{z} + 1 \right) dz = 0 ;$$

whence $-\lambda = \frac{x}{m} = \frac{y}{n} = \frac{z}{p} = \frac{a}{m+n+p}$, giving x, y, z for an extremum.

$$\text{Now since } df = f \left(\frac{m}{x} dx + \frac{n}{y} dy + \frac{p}{z} dz \right),$$

$$d^2 f = df \left(\frac{m}{x} dx + \frac{n}{y} dy + \frac{p}{z} dz \right) - \left\{ \frac{m}{x^2} (dx)^2 + \frac{n}{y^2} (dy)^2 + \frac{p}{z^2} (dz)^2 \right\} \cdot f$$

$$= - \left\{ \frac{m}{x^2} (dx)^2 + \frac{n}{y^2} (dy)^2 + \frac{p}{z^2} (dz)^2 \right\} f < 0, \text{ since } df = 0 \text{ at an}$$

extremum.

Hence the extremum is a maximum.]

12. Find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(C. H. 1962, 1963)

[Here we have to find the greatest value of $V = 8xyz$, where

x, y, z are all positive, subject to the condition $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\text{From } V = 8xyz, \text{ we get } \frac{dV}{V} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 ;$$

$$\text{and from } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ we have } \frac{x}{a^2} dx + \frac{y}{b^2} dy + \frac{z}{c^2} dz = 0.$$

$$\therefore \left(\frac{x}{a^2} + \frac{\lambda}{x} \right) dx + \left(\frac{y}{b^2} + \frac{\lambda}{y} \right) dy + \left(\frac{z}{c^2} + \frac{\lambda}{z} \right) dz = 0,$$

$$\text{whence } -\lambda = \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}. \therefore x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}.$$

Thus the extreme value of V is $\frac{8abc}{3\sqrt{3}}$.

Now $dV = \left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \right) V$,

$$\therefore d^2V = \left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \right) dV - \left\{ \left(\frac{dx}{x} \right)^2 + \left(\frac{dy}{y} \right)^2 + \left(\frac{dz}{z} \right)^2 \right\} V$$

$$= - \left\{ \left(\frac{dx}{x} \right)^2 + \left(\frac{dy}{y} \right)^2 + \left(\frac{dz}{z} \right)^2 \right\} V < 0 \text{ as } dV = 0 \text{ at an}$$

extremum. Hence $\frac{8abc}{3\sqrt{3}}$ is a maximum value.]

13. Find the maximum value of $\cos x \cos y \cos z$, where x, y, z are the angles of a plane triangle. $\left[\frac{1}{8} \right]$ (C. H. 1959)

14. Find the maximum value of the function $f(x, y, z) = x^2 y^2 z^2$ subject to the condition, $x^2 + y^2 + z^2 = c^2$. $\left[\frac{c^6}{27} \right]$ (C. H. 1960)

15. Determine the conditions which the angles x, y, z of a plane triangle should fulfil in order that

$$\sin^l x \sin^m y \sin^n z$$

may be a maximum.

(C. H. 1955)

$$\left[\frac{\tan x}{l} = \frac{\tan y}{m} = \frac{\tan z}{n} \right]$$

16. A and B are fixed points and P is a variable point on a fixed straight line; show that $\lambda AP + \mu BP$ will be a minimum if $\lambda \cos \theta = \mu \cos \phi$, θ and ϕ being the angles which AP and BP make with the fixed line. (C. H. 1953)

17. If $u = a^3 x^2 + b^3 y^2 + c^3 z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, find the maximum or minimum value of u . (C. H. 1964)

18.8. Differentiation under the sign of integration. Leibnitz's Rule.

If $f(x, \alpha)$ be a function of the two independent variables x and α , such that in the finite region ($x_0 \leq x \leq X$; $\alpha_0 \leq \alpha \leq \alpha_1$)

- (i) it is bounded,
- (ii) it is continuous with respect to both variables,
- (iii) it possesses a derivative with respect to α ,

(iv) the derivative $\frac{\partial}{\partial \alpha} f(x, \alpha)$ is continuous in the region,
then will

$$\frac{\partial}{\partial \alpha} \int_{x_0}^X f(x, \alpha) dx = \int_{x_0}^X \frac{\partial}{\partial \alpha} f(x, \alpha) dx + \frac{dX}{d\alpha} f(X, \alpha) - \frac{dx_0}{d\alpha} f(x_0, \alpha).$$

$$\text{Suppose } F(\alpha) = \int_{x_0}^X f(x, \alpha) dx,$$

$$\text{then } F(\alpha + \Delta \alpha) = \int_{x_0 + \Delta x_0}^{X + \Delta X} f(x, \alpha + \Delta \alpha) dx,$$

where it is supposed that both X and x_0 depend upon α .

$$\begin{aligned} \therefore F(\alpha + \Delta \alpha) - F(\alpha) &= \int_{x_0 + \Delta x_0}^{X + \Delta X} f(x, \alpha + \Delta \alpha) dx - \int_{x_0}^X f(x, \alpha) dx, \\ &= \int_X^{X + \Delta X} f(x, \alpha + \Delta \alpha) dx + \int_{x_0}^X f(x, \alpha + \Delta \alpha) dx \\ &\quad + \int_{x_0 + \Delta x_0}^{x_0} f(x, \alpha + \Delta \alpha) dx - \int_{x_0}^X f(x, \alpha) dx, \\ &= \int_X^{X + \Delta X} f(x, \alpha + \Delta \alpha) dx - \int_{x_0}^{x_0 + \Delta x_0} f(x, \alpha + \Delta \alpha) dx \\ &\quad + \int_{x_0}^X \{f(x, \alpha + \Delta \alpha) - f(x, \alpha)\} dx. \end{aligned}$$

Since $f(x, \alpha)$ is continuous in (x_0, X) we have by the first mean value theorem for integrals

$$\int_X^{X + \Delta X} f(x, \alpha + \Delta \alpha) dx = \Delta X f(X + \theta \Delta X, \alpha + \Delta \alpha)$$

$$\int_{x_0}^{x_0 + \Delta x_0} f(x, \alpha + \Delta \alpha) dx = \Delta x_0 f(x_0 + \theta' \Delta x_0, \alpha + \Delta \alpha),$$

where $0 < \theta < 1, 0 < \theta' < 1$.

$$\begin{aligned} \therefore F(\alpha + \Delta \alpha) - F(\alpha) &= \Delta X f(X + \theta \Delta X, \alpha + \Delta \alpha) \\ &\quad - \Delta x_0 f(x_0 + \theta' \Delta x_0, \alpha + \Delta \alpha) \\ &\quad + \int_{x_0}^X \{f(x, \alpha + \Delta \alpha) - f(x, \alpha)\} dx. \end{aligned}$$

Now by the mean value theorem of the differential calculus

$$f(x, \alpha + \Delta \alpha) - f(x, \alpha) = \Delta \alpha \frac{\partial}{\partial \alpha} f(x, \alpha + \theta'' \Delta \alpha),$$

($0 < \theta'' < 1$), if $f(x, \alpha)$ possesses a derivative with respect to α .

$$\begin{aligned} \therefore \frac{F(\alpha + \Delta \alpha) - F(\alpha)}{\Delta \alpha} &= \frac{\Delta X}{\Delta \alpha} f(X + \theta \Delta X, \alpha + \Delta \alpha) \\ &\quad - \frac{\Delta x_0}{\Delta \alpha} f(x_0 + \theta' \Delta x_0, \alpha + \Delta \alpha) \\ &\quad + \int_{x_0}^X \frac{\partial}{\partial \alpha} f(x, \alpha + \theta'' \Delta \alpha) dx. \end{aligned}$$

If further $\frac{\partial}{\partial \alpha} f(x, \alpha)$ is continuous,

$$\Delta \alpha \xrightarrow{\text{Lt}} 0 \quad \frac{\partial}{\partial \alpha} f(x, \alpha + \theta'' \Delta \alpha) = \frac{\partial}{\partial \alpha} f(x, \alpha).$$

Thus finally

$$\frac{\partial}{\partial \alpha} \int_{x_0}^X f(x, \alpha) dx = \frac{dX}{d\alpha} f(X, \alpha) - \frac{dx_0}{d\alpha} f(x_0, \alpha) + \int_{x_0}^X \frac{\partial}{\partial \alpha} f(x, \alpha) dx.$$

Cor. If the limits of integration x_0 and X be independent of α ,

$\frac{dX}{d\alpha}, \frac{dx_0}{d\alpha}$ vanish and the formula takes the simple form

$$\frac{\partial}{\partial \alpha} \int_{x_0}^X f(x, \alpha) dx = \int_{x_0}^X \frac{\partial}{\partial \alpha} f(x, \alpha) dx.$$

Examples B: On Differentiation under the sign of Integration.

1. Evaluate

$$u = \int_0^{\pi} \frac{\log(1+a \cos x)}{\cos x} dx, \quad |a| < 1.$$

Here the integrand is a continuous function of a in $(0, a)$ and of x in $(0, \pi)$; also its partial derivative with respect to a exists and is continuous in the same region. Therefore by Leibnitz's rule

$$\begin{aligned} \frac{\partial u}{\partial a} &= \int_0^{\pi} \frac{\partial}{\partial a} \left\{ \frac{\log(1+a \cos x)}{\cos x} \right\} dx = \int_0^{\pi} \frac{dx}{1+a \cos x} \\ &= \frac{2}{\sqrt{1-a^2}} \left| \tan^{-1} \sqrt{\frac{1-a}{1+a}} \tan \frac{x}{2} \right|_0^{\pi} = \frac{\pi}{\sqrt{1-a^2}}. \end{aligned}$$

$$\therefore u = \int \frac{\pi}{\sqrt{1-a^2}} da = \pi \sin^{-1} a + c.$$

Now when $a=0$, we have $u=0$ and so $c=0$. Thus

$$u = \int_0^{\pi} \frac{\log(1+a \cos x)}{\cos x} dx = \pi \sin^{-1} a.$$

2. Find the value of

$$u = \int_0^{\frac{\pi}{2}} \frac{\log(1+\cos \alpha \cos x)}{\cos x} dx.$$

The integrand is a continuous function of α in $(0, \alpha)$ and of x in $(0, \frac{\pi}{2})$; its partial derivative with respect to α exists and is continuous in the same region. Hence

$$\frac{\partial u}{\partial \alpha} = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \left\{ \frac{\log(1+\cos \alpha \cos x)}{\cos x} \right\} dx$$

$$\begin{aligned}
 &= - \int_0^{\frac{\pi}{2}} \frac{\sin \alpha}{1 + \cos \alpha \cos x} dx \\
 &= - \frac{2 \sin \alpha}{\sqrt{1 - \cos^2 \alpha}} \left| \tan^{-1} \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \tan \frac{x}{2} \right|_0^{\frac{\pi}{2}} = -\alpha.
 \end{aligned}$$

$$\therefore u = - \int \alpha d\alpha = -\frac{\alpha^2}{2} + c.$$

Now $\alpha = \frac{\pi}{2}$ gives $u = 0$ and so $c = \frac{\pi^2}{8}$. Therefore

$$u = \int_0^{\frac{\pi}{2}} \frac{\log(1 - \cos \alpha \cos x)}{\cos x} dx = \frac{1}{2} \left(\frac{\pi^2}{4} - \alpha^2 \right).$$

3. Show that

$$u = \int_0^{\pi} \frac{\log(1 + \sin \alpha \cos x)}{\cos x} dx = \pi \alpha.$$

It is easily seen that Leibnitz's rule is applicable to the integral

$$\begin{aligned}
 \therefore \frac{\partial u}{\partial \alpha} &= \int_0^{\pi} \frac{\partial}{\partial \alpha} \left\{ \frac{\log(1 + \sin \alpha \cos x)}{\cos x} \right\} dx = \int_0^{\pi} \frac{\cos \alpha}{1 + \sin \alpha \cos x} dx \\
 &= 2 \left| \tan^{-1} \sqrt{\frac{1 - \sin \alpha}{1 + \sin \alpha}} \tan \frac{x}{2} \right|_0^{\pi} = \pi.
 \end{aligned}$$

So $u = \int \pi d\alpha = \pi \alpha + c$. $\alpha = 0$ gives $u = 0$, and therefore $c = 0$.

Thus

$$u = \int_0^{\pi} \frac{\log(1 + \sin \alpha \cos x)}{\cos x} dx = \pi \alpha.$$

4. Find the value of

$$u = \int_0^{\frac{\pi}{2}} \log \left(\frac{a + b \sin \theta}{a - b \sin \theta} \right) \frac{d\theta}{\sin \theta} \quad (a > b).$$

Put $\frac{b}{a} = \alpha$, then

$$u = \int_0^{\frac{\pi}{2}} \log \left(\frac{1 + \alpha \sin \theta}{1 - \alpha \sin \theta} \right) \frac{d\theta}{\sin \theta} \quad (\alpha < 1).$$

Differentiating with respect to α , we have

$$\frac{\partial u}{\partial \alpha} = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \alpha} \left\{ \log \left(\frac{1 + \alpha \sin \theta}{1 - \alpha \sin \theta} \right) \frac{d\theta}{\sin \theta} \right\}$$

$$= \int_0^{\frac{\pi}{2}} \left\{ \frac{d\theta}{1 + \alpha \sin \theta} + \frac{d\theta}{1 - \alpha \sin \theta} \right\}.$$

$$\text{In } \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \alpha \sin \theta}, \text{ put } \theta = \frac{\pi}{2} - \phi;$$

$$\text{it is transformed into } \int_0^{\frac{\pi}{2}} \frac{d\phi}{1 + \alpha \cos \phi};$$

$$\text{again in } \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 - \alpha \sin \theta}, \text{ put } \theta = \phi - \frac{\pi}{2};$$

$$\text{it becomes } \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{1 + \alpha \cos \phi}$$

$$\therefore \frac{\partial u}{\partial \alpha} = \int_0^{\pi} \frac{d\theta}{1 + \alpha \cos \phi} = \frac{2}{\sqrt{1 - \alpha^2}} \left| \tan^{-1} \sqrt{\frac{1 - \alpha}{1 + \alpha}} \tan \frac{\phi}{2} \right|_0^{\pi} = \frac{\pi}{\sqrt{1 - \alpha^2}}$$

$$\text{So } u = \pi \int \frac{d\alpha}{\sqrt{1 - \alpha^2}} = \pi \sin^{-1} \alpha + c.$$

On making $b=0$, i.e., $\alpha=0$, we find $c=0$.

$$\therefore u = \int_0^{\frac{\pi}{2}} \log \left(\frac{a+b \sin \theta}{a-b \sin \theta} \right) \frac{d\theta}{\sin \theta} = \pi \sin^{-1} \left(\frac{b}{a} \right).$$

5. Evaluate

$$u = \int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2} dx.$$

$$\text{Deduce that } \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2.$$

In u , the upper limit is a function of α , and so by Leibnitz's rule

$$\frac{\partial u}{\partial \alpha} = \int_0^{\alpha} \frac{\partial}{\partial \alpha} \left(\frac{\log(1+\alpha x)}{1+x^2} \right) dx + 1 \cdot \frac{\log(1+\alpha^2)}{1+\alpha^2}. \quad (\text{Art. 18'6})$$

$$= \int_0^{\alpha} \frac{x}{(1+\alpha x)(1+x^2)} dx + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$\text{Now } \frac{x}{(1+\alpha x)(1+x^2)} = \frac{x+\alpha}{(1+\alpha^2)(1+x^2)} - \frac{\alpha}{(1+\alpha^2)(1+\alpha x)},$$

$$\therefore \int_0^{\alpha} \frac{x}{(1+\alpha x)(1+x^2)} dx = \left| \frac{1}{2} \frac{\log(1+x^2)}{1+\alpha^2} \right|_0^{\alpha} \\ + \left| \frac{\alpha}{1+\alpha^2} \tan^{-1} x \right|_0^{\alpha} - \left| \frac{\log(1+\alpha x)}{1+\alpha^2} \right|_0^{\alpha}$$

$$= \frac{\alpha}{1+\alpha^2} \tan^{-1} \alpha - \frac{1}{2(1+\alpha^2)} \log(1+\alpha^2).$$

Substituting,

$$\frac{\partial u}{\partial \alpha} = \frac{1}{2} \frac{\log(1+\alpha^2)}{1+\alpha^2} + \frac{\alpha}{1+\alpha^2} \tan^{-1} \alpha.$$

Integrating with respect to α ,

$$\begin{aligned} u &= \frac{1}{2} \int \frac{\log(1+\alpha^2)}{1+\alpha^2} d\alpha + \int \frac{\alpha}{1+\alpha^2} \tan^{-1} \alpha d\alpha \\ &= \frac{1}{2} \left[\log(1+\alpha^2) \tan^{-1} \alpha - \int \frac{2\alpha}{1+\alpha^2} \tan^{-1} \alpha d\alpha \right] + \int \frac{\alpha}{1+\alpha^2} \tan^{-1} \alpha d\alpha \\ &= \frac{1}{2} \log(1+\alpha^2) \tan^{-1} \alpha + c. \end{aligned}$$

Now $\alpha=0$ gives $u=0$ and so $c=0$. Thus finally

$$u = \int_0^{\alpha} \frac{\log(1+x^2)}{1+x^2} dx = \frac{1}{2} \log(1+\alpha^2) \cdot \tan^{-1} \alpha.$$

Putting $\alpha=1$ in this, we deduce

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2} dx = \frac{\pi}{8} \log 2.$$

Note. An investigation of the conditions under which Leibnitz's rule can be extended to improper integrals is beyond the scope of this elementary book. We shall therefore *assume* that the necessary conditions are satisfied by the improper integrals which occur in the examples that follow.

6. Prove that

$$u = \int_0^{\infty} e^{-\alpha x} \frac{\sin \beta x}{x} dx = \tan^{-1} \left(\frac{\beta}{\alpha} \right), \quad (C. H. 1962)$$

and deduce the value of

$$\int_0^{\infty} \frac{\sin x}{x} dx.$$

Assuming the validity of differentiation under the integral sign, we have, on differentiating with respect to β ,

$$\frac{\partial u}{\partial \beta} = \int_0^{\infty} \frac{\partial}{\partial \beta} \left(e^{-\alpha x} \frac{\sin \beta x}{x} \right) dx$$

$$= \int_0^{\infty} e^{-\alpha x} \cos \beta x \, dx = \frac{\alpha}{\alpha^2 + \beta^2}.$$

$$\therefore u = \int \frac{\alpha \, d\beta}{\alpha^2 + \beta^2} = \tan^{-1} \left(\frac{\beta}{\alpha} \right) + c.$$

Now u vanishes when $\beta=0$, this gives $c=0$. Therefore

$$\int_0^{\infty} e^{-\alpha x} \frac{\sin \beta x}{x} \, dx = \tan^{-1} \left(\frac{\beta}{\alpha} \right).$$

In this put $\beta=1$ and make α tend to 0; we then have

$$\int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$

7. Show that

$$u = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)} \, dx = \frac{\pi}{2} \log(1+a), \quad (a \geq 0).$$

Assuming that the application of Leibnitz's rule is valid, we have

$$\frac{\partial u}{\partial a} = \int_0^{\infty} \frac{\partial}{\partial a} \left\{ \frac{\tan^{-1}(ax)}{x(1+x^2)} \right\} \, dx = \int_0^{\infty} \frac{1}{(1+a^2x^2)(1+x^2)} \, dx$$

$$= \int_0^{\infty} \frac{1}{1-a^2} \left\{ \frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2} \right\} \, dx$$

$$= \frac{1}{1-a^2} \left[\tan^{-1} x \right]_0^{\infty} - \frac{a}{1-a^2} \left[\tan^{-1}(ax) \right]_0^{\infty}$$

$$= \frac{1-a}{1-a^2} \frac{\pi}{2} = \frac{\pi}{2(1+a)}.$$

$$\therefore u = \frac{\pi}{2} \int \frac{da}{1+a} = \frac{\pi}{2} \log(1+a) + c.$$

$a=0$ gives $u=0$, whence $c=0$.

$$\therefore \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a).$$

8. Prove that

$$u = \int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2} dx = \frac{\pi}{b} \log\left(\frac{a+b}{b}\right).$$

Under the usual assumption, we have

$$\begin{aligned} \frac{\partial u}{\partial a} &= \int_0^{\infty} \frac{2ax^2 dx}{(1+a^2x^2)(1+b^2x^2)} \\ &= \frac{1}{a^2-b^2} \left[\int_0^{\infty} \frac{2a dx}{1+b^2x^2} - \int_0^{\infty} \frac{2a dx}{1+a^2x^2} \right] \\ &= \frac{1}{a^2-b^2} \left(\frac{a}{b} - 1 \right) \pi = \frac{\pi}{b(a+b)}; \end{aligned}$$

$$\therefore u = \frac{\pi}{b} \int \frac{da}{a+b} = \frac{\pi}{b} \log(a+b) + c.$$

To determine c , put $a=0$, then $u=0$ so that $c = -\frac{\pi}{b} \log b$;

$$\therefore \int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2} dx = \frac{\pi}{b} \log\left(\frac{a+b}{b}\right).$$

9. Evaluate

$$\int_0^{\frac{\pi}{2}} \log(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) d\theta, (\alpha, \beta > 0).$$

$$\text{In } \int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2} dx = \frac{\pi}{b} \log\left(\frac{a+b}{b}\right),$$

put $\alpha = \frac{1}{\alpha}, b = \frac{1}{\beta}$; we then have

$$\beta^2 \int_0^{\infty} \frac{\log(\alpha^2 + x^2) - 2 \log \alpha}{\beta^2 + x^2} dx = \pi \beta \log \frac{\alpha + \beta}{\alpha}.$$

$$\begin{aligned}\therefore \beta^2 \int_0^{\infty} \frac{\log(\alpha^2 + x^2)}{\beta^2 + x^2} dx \\&= 2\beta^2 \log \alpha \int_0^{\infty} \frac{dx}{\beta^2 + x^2} + \pi\beta \log \frac{\alpha + \beta}{\alpha} \\&= \pi\beta \log(\alpha + \beta),\end{aligned}$$

$$\text{i.e., } \int_0^{\infty} \frac{\log(\alpha^2 + x^2)}{\beta^2 + x^2} dx = \frac{\pi}{\beta} \log(\alpha + \beta).$$

Next put $x = \beta \tan \theta$, then

$$\frac{\pi}{\beta} \log(\alpha + \beta) = \int_0^{\frac{\pi}{2}} \frac{\log(\alpha^2 + \beta^2 \tan^2 \theta)}{\beta} d\theta,$$

$$\begin{aligned}\text{or, } \pi \log(\alpha + \beta) &= \int_0^{\frac{\pi}{2}} \log(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) d\theta \\&\quad - \int_0^{\frac{\pi}{2}} \log \cos^2 \theta d\theta.\end{aligned}$$

$$\text{Now } \int_0^{\frac{\pi}{2}} \log \cos^2 \theta d\theta = 2 \int_0^{\frac{\pi}{2}} \log \cos \theta d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} \log \sin \theta d\theta = \pi \log \left(\frac{1}{2} \right).$$

$$\therefore \int_0^{\frac{\pi}{2}} \log(\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) d\theta$$

$$= \pi \log(\alpha + \beta) + \pi \log \left(\frac{1}{2} \right) = \pi \log \left(\frac{\alpha + \beta}{2} \right).$$

10. Show that

$$(i) \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}};$$

$$(ii) \int_0^{\infty} \frac{dx}{(x^2+a)^{n+1}} = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{1}{a^{n+\frac{1}{2}}}.$$

For (i) start with the equation

$$\int_0^{\infty} e^{-ax} dx = \frac{1}{a}$$

and differentiate n times with respect to a .

For (ii), start with

$$\int_0^{\infty} \frac{dx}{x^2+a} = \frac{\pi}{2} \cdot \frac{1}{\sqrt{a}};$$

after n differentiations with respect to a , we get the result (ii)

11. Starting with the integrals

$$\int_0^{\infty} e^{-ax} \cos mx dx = \frac{a}{a^2+m^2},$$

$$\int_0^{\infty} e^{-ax} \sin mx dx = \frac{m}{a^2+m^2},$$

prove by differentiation that

$$\int_0^{\infty} e^{-ax} x^n \cos mx dx = \frac{n! \cos(n+1)\theta}{(a^2+m^2)^{\frac{n+1}{2}}},$$

$$\int_0^{\infty} e^{-ax} x^n \sin mx dx = \frac{n! \sin(n+1)\theta}{(a^2+m^2)^{\frac{n+1}{2}}},$$

where $\theta = \tan^{-1} \left(\frac{m}{a} \right)$.

12. Prove that

$$\int_0^{\infty} \frac{\cos mx}{1+x^2} dx = \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}.$$

$$\text{Let } v = \int_0^{\infty} \frac{\sin mx}{x(1+x^2)} dx.$$

Assuming the validity of differentiation under the integral sign, we have

$$\frac{dv}{dm} = \int_0^{\infty} \frac{-\sin mx}{1+x^2} dx = u \text{ say,}$$

$$\begin{aligned} \frac{d^2 v}{dm^2} &= - \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = - \left\{ \int_0^{\infty} \frac{\sin mx}{x} dx - \int_0^{\infty} \frac{\sin mx}{x(1+x^2)} dx \right\} \\ &= -\frac{\pi}{2} + v, \end{aligned}$$

$$\text{or, } \frac{d^2 v}{dm^2} - v = -\frac{\pi}{2}.$$

$$\text{Solving, } v = Ae^m + Be^{-m} + \frac{\pi}{2}.$$

$$\text{Now } m=0 \text{ in } v \text{ gives } v=0; \therefore A+B+\frac{\pi}{2}=0 \quad \dots \quad (1)$$

$$u = \frac{dv}{dm} = Ae^m - Be^{-m}.$$

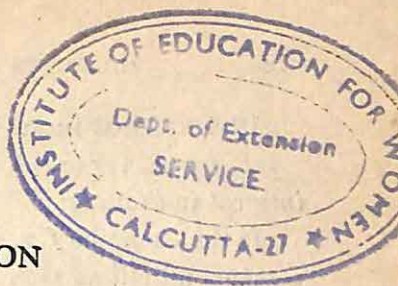
$$m=0 \text{ in } u \text{ gives } u = \frac{\pi}{2}, \text{ so that } A-B = \frac{\pi}{2}. \quad \dots \quad (2)$$

$$\text{From (1) and (2), we get } A=0, \text{ and } B = -\frac{\pi}{2}.$$

$$\therefore u = \int_0^{\infty} \frac{-\sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m},$$

$$\text{and } \frac{dv}{dm} = - \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = -\frac{\pi}{2} e^{-m},$$

$$\text{i.e., } \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}.$$



CHAPTER 19

DOUBLE INTEGRATION

19.1. Infinitesimal Region.

Let A be a given region bounded by a curve C and ω the area enclosed by C . As ω tends to zero, C may dwindle to a point or may become a finite length of a curve. When C dwindles to a point, the region A is said to be *infinitesimal in all its directions*.

19.2. Double Integration.

Let the region A of the plane, in which the bounded function $f(x, y)$ is defined, be divided in any manner into sub-regions $\alpha_1, \alpha_2, \dots, \alpha_n$ and let ω_r be the area of the sub-region α_r and M_r, m_r be the bounds of $f(x, y)$ in α_r . Consider the two sums

$$S = \sum_1^n \omega_r M_r, \quad s = \sum_1^n \omega_r m_r$$

each of which has a definite value for any particular subdivision of A . None of the sums S is less than $m\Omega$ where Ω is the area of the region A and m the lower bound of $f(x, y)$ in A ; hence these sums have a lower bound I . Likewise none of the sums s is greater than $M\Omega$ where M is the upper bound of $f(x, y)$ in A ; hence the sums s have an upper bound I' . It can moreover be shown that any of the sums S is greater than or equal to any of the sums s . Hence it follows that

$$I \geq I'.$$

If for a function $f(x, y)$, $I = I'$ i.e., S and s tend to the same limit I when the sub-regions α_r are infinitesimal in all their directions, $f(x, y)$ is said to possess a double integral extended over the region A and represented by the symbol

$$\int \int_A f(x, y) dA, \text{ or, } \int \int_A f(x, y) dx dy,$$

A being called the *field of integration* of the double integral.

19'3. Continuous Functions.

A function $f(x, y)$, continuous in a region A , possesses a double integral in that region.

If the function $f(x, y)$ is continuous, the sums S and s approach a common limit as each of the sub-regions approaches zero in all its dimensions. For suppose δ is a positive number such that the oscillation of the function is less than ϵ in any portion of A which is less than δ in all its dimensions. Given ϵ , we can find such a δ as $f(x, y)$ is a continuous function. If each of the sub-regions $\alpha_1, \alpha_2, \dots, \alpha_n$ be less than δ in all its dimensions, each of the difference $M_r - m_r$ will be less than ϵ and hence $S - s$ will be less than $\epsilon\Omega$ where Ω denotes the total area of A . But we have

$$S - s = S - I + I - I' + I' - s$$

where none of $S - I$, $I - I'$, $I' - s$ can be negative. Hence, in particular, $I - I' < \epsilon\Omega$, and as ϵ is arbitrary, it follows that $I = I'$. Moreover each of $S - I$ and $I - s$ can be made less than any pre-assigned number by a proper choice of ϵ . Hence the sums S and s have a common limit I and so $f(x, y)$ has a double integral over the field A and denoted by

$$\iint_A f(x, y) dA \text{ or } \iint_A f(x, y) dx dy.$$

19'4. Mean Value Theorem for Double Integrals.

If (ξ_r, η_r) be any point inside or on the boundary of the region α_r , it is easy to see that the sum $\Sigma f(\xi_r, \eta_r) \omega_r$ lies between the two sums S and s , or is equal to one of them. It therefore approaches the double integral as its limit whatever be the method of choice of the point (ξ_r, η_r) .

Let $f(x, y)$ be a function which is continuous in A and let $\phi(x, y)$ be another function which is continuous and has the same sign throughout A . If M and m be the bounds of $f(x, y)$ in A and we suppose for definiteness that $\phi(x, y)$ is positive in A , it is clear that

$$M\phi(\xi_r, \eta_r) \omega_r \geq f(\xi_r, \eta_r) \phi(\xi_r, \eta_r) \geq m\phi(\xi_r, \eta_r) \omega_r.$$

Adding all these inequalities for $r = 1, 2, \dots, n$ and passing to the limit, we get

$$\iint_A f(x, y) \phi(x, y) dx dy = \mu \iint_A \phi(x, y) dx dy$$

where μ lies between M and m . Since the continuous function $f(x, y)$ assumes the value μ at a point (ξ, η) inside the contour of the region A , we may write the above in the form

$$\iint_A f(x, y) \phi(x, y) dx dy = f(\xi, \eta) \iint_A \phi(x, y) dx dy,$$

which is the Mean Value Theorem for double integrals.

In particular, if $\phi(x, y) = 1$, $\iint_A dx dy = \Omega$, the area of the region A , and the above formula reduces to

$$\iint_A f(x, y) dx dy = \Omega f(\xi, \eta).$$

19.5. Geometrical meaning of a double integral.

Suppose $f(x, y)$ is defined in the region A and it possesses a double integral over the field A . Let the region A be sub-divided in any manner into sub-regions $\alpha_1, \alpha_2, \dots, \alpha_r, \dots, \alpha_n$ and let ω_r be the area of the sub-region α_r and M_r, m_r are the upper and lower bounds of $f(x, y)$ in ω_r . Let the prism, erected on ω_r as base and with sides parallel to the z -axis, cut off from the surface $z = f(x, y)$ an area σ_r . Then if V_r be the volume of this prism, we have

$$\omega_r m_r \leq V_r \leq \omega_r M_r;$$

$$\text{or, } \sum_1^n \omega_r m_r \leq \sum_1^n V_r \leq \sum_1^n \omega_r M_r.$$

Since the double integral $\iint_A f(x, y) dA$ exists, it is equal to

$$\text{Lt } \Sigma \omega_r M_r = \text{Lt } \Sigma \omega_r m_r = \Sigma V_r.$$

$$\therefore \iint_A f(x, y) dx dy = \Sigma V_r = V$$

where V is the volume of the portion of the three-dimensional space generated by the motion of a line parallel to the z -axis, moving along the contour of the region A and bounded below by the field A and above by the surface $z = f(x, y)$

19'6. Evaluation of double integrals.

Let $f(x, y)$ be a continuous function defined in the region A and having a double integral over the field A . We proceed to show that the evaluation of the double integral can be reduced to the successive evaluation of two simple integrals.

Case I. We first consider the case where the field A is a rectangle R bounded by $x=x_0, x=X$ and $y=y_0, y=Y$. Suppose this rectangle is divided by parallels to the two axes $x=x_i, y=y_k$ ($i=1, 2, \dots, n; k=1, 2, \dots, m$). The area of the small rectangle R_{ik} bounded by $x=x_{i-1}, x=x_i; y=y_{k-1}, y=y_k$ is

$$(x_i - x_{i-1})(y_k - y_{k-1}) = \omega_{ik}, \text{ say.}$$

Now if (ξ_{ik}, η_{ik}) be a point R_{ik} and M_{ik}, m_{ik} are the upper and lower bounds of $f(x, y)$ in it, we have

$$m_{ik} \omega_{ik} \leq f(\xi_{ik}, \eta_{ik}) \omega_{ik} \leq M_{ik} \omega_{ik}.$$

$$\therefore \sum_{i=1}^n \sum_{k=1}^m m_{ik} \omega_{ik} \leq \sum_{i=1}^n \sum_{k=1}^m f(\xi_{ik}, \eta_{ik}) \omega_{ik} \leq \sum_{i=1}^n \sum_{k=1}^m M_{ik} \omega_{ik}.$$

Since by supposition, the function $f(x, y)$ possesses a double integral.

$$\text{Lt} \sum_{i=1}^n \sum_{k=1}^m m_{ik} \omega_{ik} = \text{Lt} \sum_{i=1}^n \sum_{k=1}^m M_{ik} \omega_{ik} = I.$$

Therefore it is sufficient for us to calculate the limiting value of

$$\begin{aligned} S &= \sum_{i=1}^n \sum_{k=1}^m f(\xi_{ik}, \eta_{ik}) \omega_{ik} \\ &= \sum_{i=1}^n \sum_{k=1}^m f(\xi_{ik}, \eta_{ik}) (x_i - x_{i-1}) (y_k - y_{k-1}). \end{aligned}$$

Now the portion of the sum S which arises from the row of rectangles between $x=x_{i-1}$ and $x=x_i$ is

$$(x_i - x_{i-1}) [f(\xi_{i1}, \eta_{i1})(y_1 - y_0) + f(\xi_{i2}, \eta_{i2})(y_2 - y_1) + \dots + f(\xi_{ik}, \eta_{ik})(y_k - y_{k-1}) + \dots].$$

Let us choose $\xi_{i1} = \xi_{i2} = \dots = \xi_{im} = x_{i-1}$ and then choose $\eta_{i1}, \eta_{i2}, \dots$ in such a way that the sum

$$f(x_{i-1}, \eta_{i1})(y_1 - y_0) + f(x_{i-1}, \eta_{i2})(y_2 - y_1) + \dots$$

is equal to the integral $\int_{y_0}^Y f(x_{i-1}, y) dy$ where x_{i-1} is to be treated as constant.

If we do the same for each of the rows of rectangles bounded by consecutive parallels to the y -axis, we finally find that

$$S = (x_1 - x_0) \int_{y_0}^Y f(x_0, y) dy + (x_2 - x_1) \int_{y_0}^Y f(x_1, y) dy + \dots,$$

or putting
$$\phi(x) = \int_{y_0}^Y f(x, y) dy,$$

$$S = \phi(x_0)(x_1 - x_0) + \phi(x_1)(x_2 - x_1) + \dots + \phi(x_{i-1})(x_i - x_{i-1}) + \dots,$$

The function $\phi(x)$, defined by a definite integral where x is taken to be a parameter, is a continuous function of x , a result which we assume. As all the intervals $(x_i - x_{i-1})$ approach zero, the series

$$\phi(x_0)(x_1 - x_0) + \phi(x_1)(x_2 - x_1) + \dots + \phi(x_{n-1})(x_n - x_{n-1})$$

tends to the integral $\int_{x_0}^X \phi(x) dx$ as n tends to infinity.

Hence the double integral is given by

$$\int_R \int f(x, y) dx dy = \int_{x_0}^X dx \int_{y_0}^Y f(x, y) dy.$$

Thus to evaluate the double integral, $f(x, y)$ should first be integrated with respect to y between the limits y_0 and Y treating x as a constant; the resulting function which is now a function of x alone should be integrated again between the limits x_0 and X .

Cor. If we proceed in the reverse order, i.e., first evaluate the

portion of S which comes from rows of rectangles lying between consecutive parallels to the x -axis, we find the analogous formula

$$\iint_R f(x, y) dx dy = \int_{y_0}^Y dy \int_{x_0}^X f(x, y) dx.$$

It follows at once that

$$\int_{x_0}^X dx \int_{y_0}^Y f(x, y) dy = \iint_R f(x, y) dx dy = \int_{y_0}^Y dy \int_{x_0}^X f(x, y) dx.$$

This furnishes a proof of the formula for integration under the sign of integration. An essential presupposition in the proof is that the limits x_0, X, y_0, Y are constants and $f(x, y)$ is continuous in A .

Case II. We now consider the case of any field whatever. Suppose the area A is bounded by $x=x_0, x=X; y=\phi_1(x), y=\phi_2(x)$, and that any line parallel to the y -axis meets the bounding curves in two points; further $\phi_1(x)$ and $\phi_2(x)$ are continuous functions of x in (x_0, X) and $\phi_1(x) \leq \phi_2(x)$. Any other case can be reduced to this.

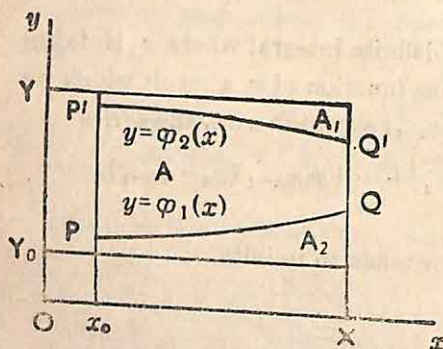


Fig. 19

Draw two lines parallel to the x -axis such that the region A is entirely included in the rectangle $(x=x_0, x=X; y=y_0, y=Y)$. Let this rectangle R be

$$R = A_1 + A + A_2$$

(see diagram).

Let a new function $\psi(x, y)$ be defined thus :

- (i) $\psi(x, y) = 0$ for any point in A_2 ,
- (ii) $\psi(x, y) = f(x, y)$ for any point in A or on its contour,
- (iii) $\psi(x, y) = 0$ for any point in A_1 .

Then

$$\iint_R \psi(x, y) dx dy = \iint_{A_1} \psi(x, y) dx dy$$

$$\begin{aligned}
 & + \int \int_A \psi(x, y) dx dy + \int \int_{A_2} \psi(x, y) dx dy \\
 & = \int \int_A f(x, y) dx dy,
 \end{aligned}$$

$$\text{or, } \int_{x_0}^X dx \int_{y_0}^Y \psi(x, y) dy = \int \int_A f(x, y) dx dy.$$

$$\text{Now } \int_{y_0}^Y \psi(x, y) dy = \int_{y_0}^{y=\phi_1^-(x)} \psi(x, y) dy + \int_{\phi_1(x)}^{\phi_2(x)} \psi(x, y) dy$$

$$+ \int_{\phi_2^-(x)}^Y \psi(x, y) dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy,$$

$$\therefore \int_{x_0}^X dx \int_{y_0}^Y \psi(x, y) dy = \int_{x_0}^X dx \int_{y_0}^{\phi_1(x)} \psi(x, y) dy$$

$$+ \int_{x_0}^X dx \int_{\phi_1^-(x)}^{\phi_2^-(x)} \psi(x, y) dy + \int_{x_0}^X dx \int_{\phi_2(x)}^Y \psi(x, y) dy$$

$$= \int_{x_0}^X dx \int_{\phi_1^-(x)}^{\phi_2^-(x)} f(x, y) dy.$$

$$\text{Hence } \int \int_A f(x, y) dx dy = \int_{x_0}^X dx \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy.$$

19.7. Triple Integrals or Volume Integrals.

Let $f(x, y, z)$ be a bounded function of three variables x, y, z , defined in the three dimensional region R [$a \leq x \leq b, c \leq y \leq d, e \leq z \leq f$].

Divide the intervals $(a, b), (c, d), (e, f)$ as under :

$$(a = x_0, x_1, \dots, x_{r-1}, x_r, \dots, x_m = b),$$

$$(c = y_0, y_1, \dots, y_{s-1}, y_s, \dots, y_n = d)$$

$$(e = z_0, z_1, \dots, z_{t-1}, z_t, \dots, z_p = f).$$

These give rise to a division of the region R into mnp sub-regions $[x_{r-1}, x_r; y_{s-1}, y_s; z_{t-1}, z_t]$, r, s, t varying from 1 to $m, 1$ to n and 1 to p respectively.

We write

$$\omega_{rst} = (x_r - x_{r-1})(y_s - y_{s-1})(z_t - z_{t-1})$$

which is the volume of the sub-region $[x_{r-1}, x_r; y_{s-1}, y_s; z_{t-1}, z_t]$.

Let M_{rst}, m_{rst} be the upper and lower bounds of $f(x, y, z)$ in the above sub-region. We write

$$S = \sum_{t=1}^p \sum_{s=1}^n \sum_{r=1}^m M_{rst} \omega_{rst},$$

$$s = \sum_{t=1}^p \sum_{s=1}^n \sum_{r=1}^m m_{rst} \omega_{rst}.$$

If (x, y, z) be any point in ω_{rst} , then

$$m_{rst} \leq f(x, y, z) \leq M_{rst};$$

$$\therefore s \leq \sum_{t=1}^p \sum_{s=1}^n \sum_{r=1}^m f(x, y, z) \omega_{rst} \leq S.$$

The lower bound of the sums S and the upper bound of the sums s are respectively called the upper and lower integrals of $f(x, y, z)$ over R and denoted by

$$\overline{\int \int \int_R f(x, y, z) dx dy dz}, \quad \underline{\int \int \int_R f(x, y, z) dx dy dz}.$$

If these two are equal, $f(x, y, z)$ is said to be integrable over R and the common value is expressed as

$$\int \int \int_R f(x, y, z) dx dy dz.$$

Cor. If $f(x, y, z)$ is continuous over R , we have

$$\int \int \int_R f(x, y, z) dx dy dz = \int_e^f dz \int_c^d dy \int_a^b f(x, y, z) dx$$

in which the order of integration with respect to x, y, z may be interchanged in any manner we please.

Examples.

1. Evaluate :

$$(a) \int \int_R \frac{xy}{ab} dx dy \text{ over } (x_0 \leq x \leq X, y_0 \leq y \leq Y).$$

$$\left[\frac{(X^2 - x_0^2)(Y^2 - y_0^2)}{4a} \right].$$

$$(b) \int \int_R \sin(x+y) \, dx \, dy \text{ over } \left(0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}\right). \quad [2]$$

$$(c) \int_0^a dx \int_0^b xy(x^2 - y^2) \, dy. \quad \left[\frac{1}{8} a^2 b^2 (a^2 - b^2)\right]$$

$$(d) \int_0^\pi \int_0^\pi \cos(x+y) \, dy \, dx. \quad [-4]$$

$$(e) \int_1^e \int_1^2 \frac{1}{xy} \, dy \, dx. \quad [\log 2]$$

$$(f) \int_0^a \int_0^b x e^{xy} \, dy \, dx. \quad \left[\frac{e^{ab}}{b} - \frac{1}{b} - a\right]$$

$$(g) \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx. \quad \left[\frac{\pi}{16}\right]$$

$$(h) \int_0^2 \int_0^{2-x} y \, dy \, dx. \quad \left[\frac{4}{3}\right]$$

2. Evaluate $\int \int x^2 y^2 \, dx \, dy$

$$(i) \text{ over the region } x \geq 0, y \geq 0, x^2 + y^2 \leq 1; \quad \left[\frac{\pi}{96}\right]$$

$$(ii) \text{ over the circle } x^2 + y^2 \leq 1. \quad (C.H. 1964) \quad \left[\frac{\pi}{24}\right]$$

3. If a continuous function $f(x, y)$, $a \leq x \leq b$, $c \leq y \leq d$, be of the form $f(x, y) = \phi(x) \cdot \psi(y)$, then prove that

$$\int \int_R f(x, y) \, dx \, dy = \int_a^b \phi(x) \, dx \cdot \int_c^d \psi(y) \, dy.$$

(C.H. 1960)

4. Show that

$$(a) \int_b^a \int_0^b \int_a^{2a} x^2 y^2 z \, dz \, dy \, dx = \frac{1}{6} a^2 b^3 (a^3 - b^3).$$

$$(b) \int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz \, dy \, dx = \frac{a^9}{90}.$$

$$(c) \int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy = \frac{4}{35}.$$

$$(d) \int_1^2 \int_0^z \int_0^{x\sqrt{3}} \frac{x}{x^2 + y^2} \, dy \, dx \, dz = \frac{\pi}{2}.$$

$$(e) \int_0^1 \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx = \frac{e^4}{8} - \frac{3}{4}e^2 + e - \frac{3}{8}.$$

5. Show that the volume between the xy plane and the paraboloid $z = 2 - x^2 - y^2$ is 2π , (i) by double integration, and also (ii) by triple integration.

$$\begin{aligned} \left[\text{Volume} &= \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (2 - x^2 - y^2) \, dy \right. \\ &\quad \left. = 4 \int_0^{\sqrt{2}} dx \int_0^{\sqrt{2-x^2}} dy \int_0^{2-x^2-y^2} dz. \right] \end{aligned}$$

6. Show that the volume bounded by the elliptic paraboloid $4z = 16 - 4x^2 - y^2$ and the XOY plane is 16π .

7. Evaluate $\int_0^1 dx \int_0^1 \frac{x^2 - y^2}{x^2 + y^2} \, dy$ and show that $\int_0^1 dy \int_0^1 \frac{x^2 - y^2}{x^2 + y^2} \, dx$

has the same value.

8. Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} \, dy = \frac{1}{2}$, but that $\int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} \, dx$

$$= -\frac{1}{2}. \text{ (Hardy)}$$

[If $x \neq 0$, $\frac{x-y}{(x+y)^3}$ is a bounded function of y and if $x=0$, then $y=0$ is a point of infinite discontinuity.

$$\begin{aligned} \text{If } x \neq 0, \phi(x) &= \int_0^1 \frac{x-y}{(x+y)^3} dy = \int_0^1 \left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy \\ &= \left| \frac{y}{(x+y)^2} \right|_0^1 = \frac{1}{(1+x)^2}; \text{ and } \phi(0) \text{ does not exist.} \end{aligned}$$

$$\begin{aligned} \text{Now } \int_0^1 \phi(x) dx &= \lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 \frac{dx}{(1+x)^2} = \lim_{\epsilon \rightarrow 0} \left| -\frac{1}{1+x} \right|_{\epsilon}^1 \\ &= \lim_{\epsilon \rightarrow 0} \left(\frac{1}{1+\epsilon} - \frac{1}{2} \right) = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \text{Again if } y \neq 0, \psi(y) &= \int_0^1 \frac{x-y}{(x+y)^3} dx = \int_0^1 \left\{ \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right\} dx \\ &= -\frac{1}{(1+y)^2} \text{ and } \psi(0) \text{ does not exist.} \end{aligned}$$

$$\text{As above } \int_0^1 \psi(y) dy = -\frac{1}{2}. \quad]$$

9. Show that

$$\int_0^1 dx \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \neq \int_0^1 dy \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx.$$

$$\begin{aligned} \left[\text{Note that } \frac{d}{dy} \left(\frac{y}{x^2 + y^2} \right) &= \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2}; \right. \\ \text{and } \frac{d}{dx} \left(\frac{x}{x^2 + y^2} \right) &= \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2}. \quad \left. \right] \end{aligned}$$

10. Calculate by double integration the portion of the area above the x -axis bounded by the semi-cubical parabola $y^2 = x^3$ and the straight line $y = x$.

[The points where $y=x$ meets $y^2=x^3$ are given by

$$x^2 = x^3, \text{ or } x^2(1-x)=0, \\ \text{i.e., } x=0, x=1.$$

$x=0$ corresponds to the origin, $x=1$ gives the abscissa of P . Thus P is the point $(1, 1)$.

$$\text{Area} = \iint_R dx dy \text{ where } R$$

is the shaded portion. For any x , y varies from the arc OP to the line OP , i.e., from $x^{\frac{3}{2}}$ to x and x varies from

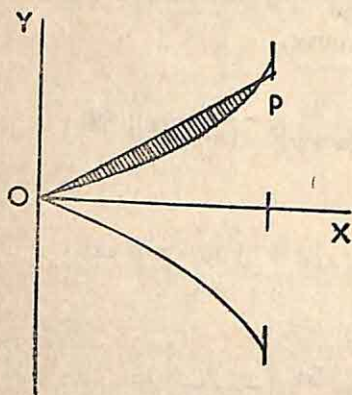


Fig. 20

0 to 1.

$$\therefore \text{Area} = \int_0^1 dx \int_{x^{\frac{3}{2}}}^x dy = \int_0^1 \left(x - x^{\frac{3}{2}} \right) dx = \frac{1}{10} \text{ sq. units. }]$$

11. Show that the area in the first quadrant bounded by the x -axis and the curves $x^2+y^2=10$, $y^2=9x$ is $\frac{1}{2} + 5 \sin^{-1} \frac{3}{\sqrt{10}}$.

[The coordinates of the point P where $y^2=9x$ meets $x^2+y^2=10$ are easily found to be $(1, 3)$.

The area required is

$$\iint_R dx dy,$$

R being the shaded portion. If we first integrate with respect to x , x varies from the parabola to the circle, i.e., from $\frac{y^2}{9}$ to $\sqrt{10-y^2}$; and y varies from 0 to 3.

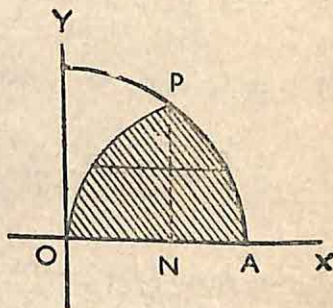


Fig. 21

$$\text{Required area} = \int_0^3 dy \int_{\frac{y^2}{9}}^{\sqrt{10-y^2}} dx = \int_0^3 \left\{ \sqrt{10-y^2} - \frac{y^2}{9} \right\} dy$$

$$= \left| \frac{y\sqrt{10-y^2}}{2} + 5 \sin^{-1} \frac{y}{\sqrt{10}} - \frac{y^3}{27} \right|_0^8$$

$$= \frac{3}{2} + 5 \sin^{-1} \frac{3}{\sqrt{10}} - 1 = \frac{1}{2} + 5 \sin^{-1} \frac{3}{\sqrt{10}}.$$

If we integrate first with respect to y , the area required is the sum of the areas OPN and NPA ; and so

$$\text{Area required} = \int_0^1 dx \int_0^{3\sqrt{x}} dy + \int_1^{\sqrt{10}} dx \int_0^{\sqrt{10-x^2}} dy$$

giving the same result as before.]

12. Find the area bounded by the following curves :

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}, \quad x + y = a.$$

$\left[x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}} \right]$ is a parabola touching the axes of x and y at $P(a, 0)$ and $Q(0, a)$ respectively.

The area in question is

$$\iint_R dx dy$$

where R is the shaded portion.

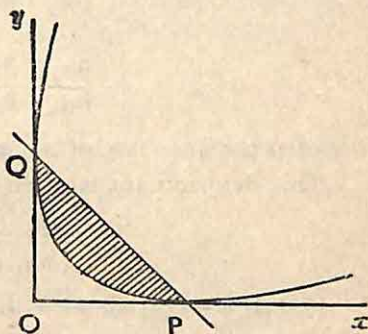


Fig. 22

For any x , y varies from the parabola to the line PQ , i.e., from $(a^{\frac{1}{2}} - x^{\frac{1}{2}})^2$ to $a - x$ whereas x varies from 0 to a .

$$\therefore \text{Area required} = \int_0^a dx \int_{(a^{\frac{1}{2}} - x^{\frac{1}{2}})^2}^{a-x} dy$$

$$= \int_0^a \left\{ (a-x) - (a^{\frac{1}{2}} - x^{\frac{1}{2}})^2 \right\} dx$$

$$= \int_0^a \{ a - x - (a + x) + 2\sqrt{ax} \} dx$$

$$= \int_0^a (2\sqrt{ax} - 2x) dx = 2 \left| \frac{2\sqrt{a}}{3} x^{\frac{3}{2}} - \frac{x^2}{2} \right|_0^a$$

$$= 2 \left(\frac{2}{3} a^2 - \frac{a^2}{2} \right) = \frac{a^2}{3}.$$

CHAPTER 20

EVALUATION OF DOUBLE INTEGRALS

20 1. Jacobians.

(1) If $x_1, x_2, \dots, x_n$ be functions of $u_1, u_2, \dots, u_n$, the determinant

$$\begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \dots & \frac{\partial x_n}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \dots & \frac{\partial x_n}{\partial u_2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial x_1}{\partial u_n} & \frac{\partial x_2}{\partial u_n} & \dots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

is called the Jacobian of $x_1, x_2, \dots, x_n$ with regard to $u_1, u_2, \dots, u_n$.

This determinant is often denoted by

$$\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(u_1, u_2, \dots, u_n)} \text{ or shortly by } J.$$

(2) If $x=f(u, v)$, $y=\phi(u, v)$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}.$$

(3) If $x=f(\xi, \eta, \zeta)$, $y=\phi(\xi, \eta, \zeta)$, $z=\psi(\xi, \eta, \zeta)$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)}.$$

20 2. Change of variables (A Lemma without proof)

(1) Let (x, y) be a point in a region A bounded by a contour C , and let x, y be transformed by the substitutions

$$x=\phi(u, v), y=\psi(u, v). \quad \dots \quad (1)$$

Corresponding to every pair of values of x, y , there are definite values of u, v , so that as x and y are confined to the region A in the x - y plane, u and v are confined to a region A' in the u - v plane, bounded by a contour C' .

In this connection we assume without proof the following lemma :

We suppose that

(i) the two functions x and y of u and v possess continuous first order partial derivatives at all points of A' and C' ;

(ii) the equations (1) transform the region A' bounded by C' into a region A of the x - y plane bounded by a curve C in such a way that a *one-to-one* correspondence exists between points in the two regions and their contours ;

(iii) the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ does not change sign at any point of A' , though it may vanish at certain points of C' ,

then

$$\iint_A f(x, y) \, dx \, dy = \iint_{A'} f\{\phi(u, v), \psi(u, v)\} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv.$$

(2) This method of transformation is also applicable to a triple integral. If the triple integral $\iiint_A F(x, y, z) \, dx \, dy \, dz$ be transformed by the substitutions

$$x = f(\xi, \eta, \zeta), \quad y = \phi(\xi, \eta, \zeta), \quad z = \psi(\xi, \eta, \zeta),$$

then

$$\begin{aligned} & \iiint_A F(x, y, z) \, dx \, dy \, dz \\ &= \iiint_{A'} F\{f(\xi, \eta, \zeta), \phi(\xi, \eta, \zeta), \psi(\xi, \eta, \zeta)\} \left| \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} \right| \, d\xi \, d\eta \, d\zeta. \end{aligned}$$

20.3. Evaluation of double and triple integrals by change of variables.

The method will be best illustrated by the examples that follow.

Examples A.

1. Evaluate the integral $\iint \sqrt{4a^2 - x^2 - y^2} \, dx \, dy$ taken over the upper half of the circle $x^2 + y^2 - 2ax = 0$. (C. H. 1966)

[Transform the integral by the substitutions $x = r \cos \theta$, $y = r \sin \theta$.

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r.$$

The field A is $x^2 + y^2 - 2ax = 0$; it is transformed into A' where A' is $r = 2a \cos \theta$, which is the same circle as A . In the upper half of this, for any θ , r varies from 0 to $2a \cos \theta$; and θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \iint_A \sqrt{4a^2 - x^2 - y^2} \, dx \, dy = \iint_{A'} \sqrt{4a^2 - r^2} \, r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2a \cos \theta} \sqrt{4a^2 - r^2} \, r \, dr$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} \left[(4a^2 - r^2)^{\frac{3}{2}} \right]_0^{2a \cos \theta} d\theta = \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^3 \theta) \, d\theta$$

$$= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} (1 - \sin \theta + \sin \theta \cos^2 \theta) \, d\theta$$

$$= \frac{8a^3}{3} \left[\theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\frac{\pi}{2}} = \frac{8a^3}{3} \left(\frac{\pi}{2} - 1 + \frac{1}{3} \right) = \frac{4}{9} (3\pi - 4) a^3.]$$

2. Evaluate $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) \, dx \, dy$ where R consists of points

in the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (C. H. 1963)

$$[\text{Put } x=a\xi, y=b\eta. \quad J = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.]$$

R is transformed into R' where R' is the positive quadrant of the circle $\xi^2 + \eta^2 = 1$. In R' , ξ varies from 0 to 1 and η from 0 to $\sqrt{1-\xi^2}$.

$$\begin{aligned} \therefore \int \int_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx dy &= \int \int_{R'} (1 - \xi^2 - \eta^2) ab d\xi d\eta \\ &= ab \int_0^1 d\xi \int_0^{\sqrt{1-\xi^2}} (1 - \xi^2 - \eta^2) d\eta \\ &= ab \int_0^1 \left(1 - \xi^2\right) \eta - \frac{\eta^3}{3} \Big|_0^{\sqrt{1-\xi^2}} d\xi = ab \int_0^1 \left\{ (1 - \xi^2)^{\frac{3}{2}} - \frac{(1 - \xi^2)^{\frac{5}{2}}}{3} \right\} d\xi \\ &= \frac{2}{3} ab \int_0^{\frac{\pi}{2}} (1 - \xi^2)^{\frac{3}{2}} d\xi = \frac{1}{8} \pi ab \text{ (on putting } \xi = \sin \theta). \end{aligned}$$

3. Evaluate $\int \int [2a^2 - 2a(x+y) - x^2 + y^2] dx dy$, the region of integration being the circle $x^2 + y^2 + 2a(x+y) = 2a^2$.
(C.H. 1962, 1963)

[The equation of the circle can be expressed as $(x+a)^2 + (y+a)^2 = 4a^2$; on putting $x = \xi - a$, $y = \eta - a$, it is transformed into $\xi^2 + \eta^2 = (2a)^2$. $|J|$ for the substitutions is 1.

$$\begin{aligned} \therefore \int \int_A [2a^2 - 2a(x+y) - (x^2 + y^2)] dx dy \\ &= \int \int_{A'} (4a^2 - \xi^2 - \eta^2) d\xi d\eta \\ &= 4 \int_0^{2a} d\xi \int_0^{\sqrt{4a^2 - \xi^2}} (4a^2 - \xi^2 - \eta^2) d\eta, \end{aligned}$$

(for as the integrand is symmetrical in ξ, η , the integral over the whole circle is 4 times the integral over a quadrant),

$$= \frac{8}{3} \int_0^{2a} (4a^2 - \xi^2)^{\frac{3}{2}} d\xi = \frac{128}{3} a^4 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta \text{ (on putting } \xi = 2a \sin \theta),$$

$$= 8\pi a^4.$$

We may also put $x+a=r \cos \theta$, $y+a=r \sin \theta$.]

4. Evaluate $\iint \frac{dx dy}{(1+x^2+y^2)^2}$

(a) over a loop of the lemniscate $(x^2+y^2)^2 - (x^2-y^2) = 0$;

(b) over a triangle whose vertices are $(0, 0)$, $(2, 0)$, $(1, \sqrt{3})$.

(C. H. 1962)

[(a) Changing to polars, we have to evaluate $\iint \frac{r dr d\theta}{(1+r^2)^2}$

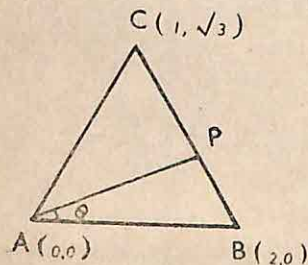
over a loop of $r^2 = \cos 2\theta$, since $|J| = r$. For half the loop, θ varies from 0 to $\frac{\pi}{4}$, and for any θ , r varies from 0 to $\sqrt{\cos 2\theta}$.

$$\therefore \iint \frac{dx dy}{(1+x^2+y^2)^2} = 2 \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{\cos 2\theta}} \frac{r dr}{(1+r^2)^2}$$

on account of symmetry,

$$= \int_0^{\frac{\pi}{4}} \left[-\frac{1}{1+r^2} \right]_0^{\sqrt{\cos 2\theta}} d\theta = \int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{1+\cos 2\theta} \right) d\theta$$

$$= \left[\theta - \frac{1}{2} \tan \theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}.$$



(b) The triangle which is the field of integration is unchanged by transformation to polars. In the triangle which is equilateral with side 2, θ varies from 0 to $\frac{\pi}{3}$ and r varies from 0 to AP .

Now

$$\frac{AP}{\sin C} = \frac{AC}{\sin APC}, \text{ or } \frac{AP}{\sin \frac{\pi}{3}} = \frac{2}{\sin(\theta + \frac{\pi}{3})}.$$

$$\therefore AP = \frac{2 \sin \frac{\pi}{3}}{\sin \left(\theta + \frac{\pi}{3} \right)} = \frac{\sqrt{3}}{\sin \phi}, \text{ where } \phi = \theta + \frac{\pi}{3},$$

$$\text{The integral} = \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} d\phi \int_0^{\frac{\sqrt{3}}{\sin \phi}} \frac{2r dr}{(1+r^2)^2}$$

$$= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left[-\frac{1}{1+r^2} \right]_0^{\frac{\sqrt{3}}{\sin \phi}} d\phi.$$

$$= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left(1 - \frac{1}{1 + \frac{3}{\sin^2 \phi}} \right) d\phi$$

$$= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{3}{3 + \sin^2 \phi} d\phi = \frac{3}{2} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{d\phi}{4 - \cos^2 \phi}$$

$$= \frac{3}{8} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left\{ \frac{1}{2 + \cos \phi} + \frac{1}{2 - \cos \phi} \right\} d\phi$$

$$= \frac{3}{8} \left[\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \tan \frac{\phi}{2} \right) + \frac{2}{\sqrt{3}} \tan^{-1} \left(\sqrt{3} \tan \frac{\phi}{2} \right) \right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$= \frac{\sqrt{3}}{4} \left\{ \tan^{-1} 1 + \tan^{-1} 3 - \tan^{-1} \frac{1}{3} - \tan^{-1} 1 \right\}$$

$$= \frac{\sqrt{3}}{4} \tan^{-1} \frac{3 - \frac{1}{3}}{1 + 1} = \frac{\sqrt{3}}{4} \tan^{-1} \frac{4}{3} = \frac{\sqrt{3}}{2} \tan^{-1} \frac{1}{2}.$$

5. Show that $\iint \sqrt{[x(2a-x) + y(2b-y)]} dx dy$ taken over the circle $x^2 + y^2 - 2ax - 2by = 0$ is $\frac{2\pi}{3} (a^2 + b^2)^{\frac{3}{2}}$.

6. Evaluate $\int \int \frac{\sqrt{a^2 b^2 - b^2 x^2 - a^2 y^2}}{\sqrt{a^2 b^2 + b^2 x^2 + a^2 y^2}} dx dy,$

the area of integration being the positive quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (C. H. 1961, '68)$$

$$\left[\frac{\pi}{8} (\pi - 2) ab \right].$$

7. Evaluate $\int \int e^{\frac{y-x}{y+x}} dx dy$ over a triangle with vertices (0, 0), (0, 1), (1, 0).

$$\left[\frac{1}{4} \left(e - \frac{1}{e} \right) \right].$$

8. Evaluate $\int \int (x^2 + y^2) dx dy$ over the region enclosed by the triangle having its vertices at (0, 0), (1, 0), (1, 1). (C. H. 1965)

$$\left[\frac{1}{3} \right].$$

9. Evaluate $\int \int \int xyz dx dy dz$ over the region defined by

$$x \geq 0, y \geq 0, z \geq 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

[Putting $x = a\xi$, $y = b\eta$, $z = c\zeta$, the field of integration is transformed into the positive octant of the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$. Also $|J| = abc$.

$\therefore$ the integral (I) is transformed into

$$\int \int \int a\xi b\eta c\zeta abc d\xi d\eta d\zeta = a^2 b^2 c^2 \int \int \int \xi \eta \zeta d\xi d\eta d\zeta.$$

In the positive octant of the sphere ζ varies from 0 to $\sqrt{1 - \xi^2 - \eta^2}$.

$$\begin{aligned} \therefore I &= a^2 b^2 c^2 \int \int \xi \eta d\xi d\eta \int_0^{\sqrt{1 - \xi^2 - \eta^2}} \zeta d\zeta \\ &= \frac{1}{2} a^2 b^2 c^2 \int \int \xi \eta (1 - \xi^2 - \eta^2) d\xi d\eta. \end{aligned}$$

Transforming to polars,

$$I = \frac{1}{2} a^2 b^2 c^2 \int \int r^2 \sin \theta \cos \theta (1 - r^2) r dr d\theta.$$

To cover the field, r must vary from 0 to 1 and θ from 0 to $\frac{\pi}{2}$.

$$\begin{aligned}\therefore I &= \frac{1}{2} a^2 b^2 c^2 \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta \int_0^1 (r^3 - r^5) dr \\ &= \frac{a^2 b^2 c^2}{48} \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = \frac{a^2 b^2 c^2}{48}.\end{aligned}$$

Note. We might also use the substitutions

$$x = ar \sin \theta \cos \phi, y = br \sin \theta \sin \phi, z = cr \cos \theta.]$$

10. Prove that $\iiint \frac{dx dy dz}{\sqrt{(1-x^2-y^2-z^2)}} = \frac{\pi^2}{8}$, the integral being taken over all positive values of x, y, z for which the expression is real.

[Transform the integral by the substitutions

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta.$$

$|J| = r^2 \sin \theta$. The field of integration is $x \geq 0, y \geq 0, z \geq 0$, and $x^2 + y^2 + z^2 \leq 1$, i.e., the positive octant of the sphere $x^2 + y^2 + z^2 = 1$. In the transformed field, r varies from 0 to 1, θ from 0 to $\frac{\pi}{2}$, ϕ from 0 to $\frac{\pi}{2}$.

$$\therefore \iiint \frac{dx dy dz}{\sqrt{(1-x^2-y^2-z^2)}} = \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_0^1 \frac{r^2 dr}{\sqrt{1-r^2}}.$$

$$\text{Now } \int_0^1 \frac{r^2 dr}{\sqrt{1-r^2}} = \int_0^{\frac{\pi}{2}} \sin^2 u du = \frac{\pi}{4} \text{ (by putting } r = \sin u \text{);}$$

$$\therefore \left[\iiint \frac{dx dy dz}{\sqrt{(1-x^2-y^2-z^2)}} = \frac{\pi^2}{8} \right]$$

11. Evaluate $\iiint \sqrt{(a^2 b^2 c^2 - b^2 c^2 x^2 - c^2 a^2 y^2 - a^2 b^2 z^2)} dx dy dz$ taken throughout the region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

[Transform by the substitutions

$$x = ar \sin \theta \cos \phi, y = br \sin \theta \sin \phi, z = cr \cos \theta.$$

In the transformed field, r varies from 0 to 1, θ from 0 to π and ϕ from 0 to 2π . The value of the integral is $\frac{1}{4} a^2 b^2 c^2 \pi^2$.]

12. Evaluate $\iiint (x^2 + y^2 + z^2) xyz \, dx \, dy \, dz$ taken throughout the sphere $x^2 + y^2 + z^2 \leq 1$. [0] (C. H. 1964)

13. Show that $\iiint e^{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}} \, dx \, dy \, dz$ taken throughout the region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ is $4\pi abc (e - 2)$.

14. Find the volume of the solid bounded by the paraboloid of revolution $x^2 + y^2 = az$, the x - y plane and the cylinder $x^2 + y^2 = 2ax$.

[$V = \iiint dz \, dy \, dx$. The limits of z are 0 and $\frac{x^2 + y^2}{a}$.

$\therefore V = \iint_R \frac{x^2 + y^2}{a} \, dy \, dx$ where R is the area of the circle $x^2 + y^2 = 2ax$. Changing to polars, we have, on account of symmetry,

$$V = 2 \int_0^\pi d\theta \int_0^{2a \cos \theta} \frac{r^2}{a} r \, dr = \frac{3}{2} \pi a^3.]$$

15. Show that the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$ is $\left(\frac{2\pi}{3} - \frac{8}{9}\right) a^3$.

16. Show that the volume included between the elliptic paraboloid $2z = x^2 + y^2$, the cylinder $x^2 + y^2 = a^2$ and the x - y plane is $\frac{\pi}{4} a^4$.

17. The sphere $x^2 + y^2 + z^2 = a^2$ is pierced by the cylinder $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. Find the volume of the portion of the sphere inside the cylinder. (C. H. 1961)

$$\left[\frac{2a^3}{9} (3\pi + 20 - 16\sqrt{2}). \right]$$

18. Show that the volume included between the sphere $x^2 + y^2 + z^2 = a^2$, the x - y plane and the cylinder $x^2(x^2 + y^2) - a^2(x^2 - y^2) = 0$ is $\frac{2a^3}{3} \left(\frac{\pi}{2} - 1 + \log 2 \right)$.

20.4. Change in the order of integration.

We have seen (Art. 19.6, Cor.) that for a continuous function if the limits are constants, the integration may be performed in any order. When the limits are not constants, we may change the order of integration provided new limits are chosen to cover the entire field of integration. This is illustrated in the examples that follow. It will be seen that for this purpose, the field has sometimes to be divided into suitable parts.

Examples B.

1. Change the order of integration in

$$\int_0^a dx \int_0^x \frac{f'(y) dy}{\sqrt{(a-x)(x-y)}}$$

and hence show that it has the value $\pi \{f(a) - f(0)\}$.

$$\left[\text{In } U = \int_0^a dx \int_0^x \frac{f'(y) dy}{\sqrt{(a-x)(x-y)}}, y \text{ varies from } y=0 \text{ to } y=x; \right.$$

and x varies from $x=0$ to $x=a$. The field is thus the triangle AOB where $OA=AB=a$. In it for any $x=ON$, y varies from 0 to NP , i.e., from 0 to x and x varies from 0 to a . Reversing the order, for any y , x varies from HM to HQ , i.e., from y to a and y varies from 0 to a .

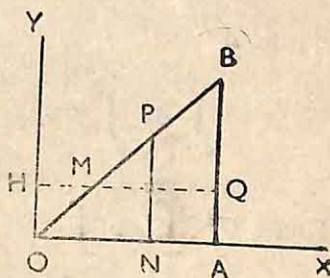


Fig. 24

$$\begin{aligned} \therefore U &= \int_0^a dx \int_0^x \frac{f'(y) dy}{\sqrt{(a-x)(x-y)}} = \int_0^a dy \int_y^a \frac{f'(y) dx}{\sqrt{(a-x)(x-y)}} \\ &= \int_0^a f'(y) dy \int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}} \end{aligned}$$

Now
$$\int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}} = 2 \left| \sin^{-1} \sqrt{\frac{x-y}{a-y}} \right|_y^a = 2 \sin^{-1} 1 = \pi.$$

$$\therefore U = \pi \int_0^a f'(y) dy = \pi \{f(a) - f(0)\}.$$

2. Prove that

$$\int_0^{2a} dx \int_0^{\sqrt{(2ax-x^2)}} f(x, y) dy = \int_0^a dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x, y) dx;$$

and hence find the value of

$$U = \int_0^{2a} dx \int_0^{\sqrt{(2ax-x^2)}} \frac{\phi'(y)(x^2+y^2)x}{\sqrt{4a^2x^2-(x^2+y^2)^2}} dy.$$

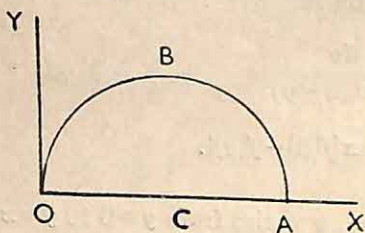


Fig. 25

[The field of integration is the area bounded by $x^2+y^2=2ax$, $y=0$, $x=0$, $x=2a$; it is the area covered by the semi-circle AOB . Now from $x^2+y^2=2ax$, we get $x=a \pm \sqrt{a^2-y^2}$; thus for any y , x varies from $a-\sqrt{a^2-y^2}$ to $a+\sqrt{a^2-y^2}$ and y varies

from 0 to a .

$$\therefore \int_0^{2a} dx \int_0^{\sqrt{(2ax-x^2)}} f(x, y) dy = \int_0^a dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x, y) dx.$$

Hence
$$U = \int_0^{2a} dx \int_0^{\sqrt{(2ax-x^2)}} \frac{\phi'(y)(x^2+y^2)x}{\sqrt{4a^2x^2-(x^2+y^2)^2}} dy$$

$$= \int_0^a \phi'(y) dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \frac{(x^2+y^2)x dx}{\sqrt{4a^2x^2-(x^2+y^2)^2}}.$$

To evaluate
$$\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \frac{(x^2+y^2)x dx}{\sqrt{4a^2x^2-(x^2+y^2)^2}}$$

put $x^2+y^2=z$, where y is kept fixed.

$$\begin{aligned}
 \text{Now } \int \frac{(x^2+y^2)x \, dx}{\sqrt{\{4a^2x^2-(x^2+y^2)^2\}}} &= \frac{1}{2} \int \frac{z \, dz}{\sqrt{\{4a^2(a^2-y^2)-(z-2a^2)^2\}}} \\
 &= -\frac{1}{4} \int \frac{-2(z-2a^2)-4a^2}{\sqrt{\{4a^2(a^2-y^2)-(z-2a^2)^2\}}} dz \\
 &= -\frac{1}{2} \left\{ \sqrt{4a^2(a^2-y^2)-(x^2+y^2-2a^2)^2} + a^2 \sin^{-1} \frac{x^2+y^2-2a^2}{2a \sqrt{a^2-y^2}} \right\} \\
 &= I_1 + I_2 \text{ say.}
 \end{aligned}$$

Also I_1 vanishes for both limits and $I_2 = a^2 \sin^{-1} \frac{x-a}{\sqrt{a^2-y^2}}$.

$$\therefore \left| a^2 \sin^{-1} \frac{x-a}{\sqrt{a^2-y^2}} \right|_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} = \pi a^2.$$

$$\text{Thus } U = \pi a^2 \int_0^a \phi'(y) \, dy = \pi a^2 \{ \phi(a) - \phi(0) \}. \quad]$$

3. Change the order of integration in the integral

$$U = \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{dy}{(1+e^y) \sqrt{1-x^2-y^2}}$$

and find the value of U .

[The field is the positive quadrant of the unit circle,

$$\begin{aligned}
 \therefore \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{dy}{(1+e^y) \sqrt{1-x^2-y^2}} \\
 = \int_0^1 \frac{dy}{1+e^y} \int_0^{\sqrt{1-y^2}} \frac{dx}{\sqrt{1-y^2-x^2}}
 \end{aligned}$$

The value of U is easily found to be $\frac{\pi}{2} \log \frac{2e}{1+e}. \quad]$

$$4. \text{ Prove that } \int_0^1 dx \int_x^1 \frac{y \, dy}{(1+xy)^2(1+y^2)} = \frac{\pi-1}{4}. \quad (C. H. 1965)$$

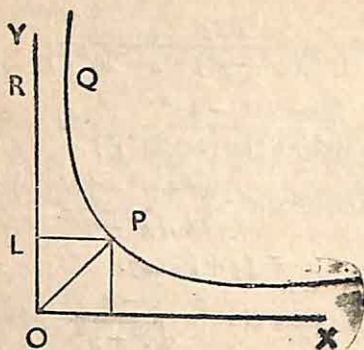


Fig. 26

[The field of integration is the area bounded by $y=x$, $xy=1$, $x=0$, $x=1$. It is the infinite region $ROPQ$ where $P(1, 1)$ is the point where $xy=1$ and $x=y$ meet. Now area $ROPQ = \Delta_1 + \Delta_2$, where Δ_1 is the triangle LOP and Δ_2 is the infinite region $RLPQ$.

In Δ_1 , for any y , x varies from 0 to y and y varies from 0 to 1. In Δ_2 for constant y , x varies from 0 to $\frac{1}{y}$ and y varies from 1 to ∞ .

$$\therefore \int_0^1 dx \int_x^{\frac{1}{x}} \frac{y \, dy}{(1+xy)^2(1+y^2)} = \int_0^1 \frac{y \, dy}{1+y^2} \int_0^y \frac{dx}{(1+xy)^2} \\ + \int_1^{\infty} \frac{y \, dy}{1+y^2} \int_0^{\frac{1}{y}} \frac{dx}{(1+xy)^2} = U_1 + U_2 \text{ say.}$$

$$U_1 = \int_0^1 \frac{y \, dy}{1+y^2} \int_0^y \frac{dx}{(1+xy)^2} = \int_0^1 \frac{1}{1+y^2} \left| -\frac{1}{1+xy} \right|_0^y dy \\ = \int_0^1 \frac{y^2}{(1+y^2)^2} dy = \frac{\pi}{8} - \frac{1}{4} \text{ (put } y = \tan \theta \text{).}$$

$$U_2 = \int_1^{\infty} \frac{dy}{1+y^2} \left| -\frac{1}{1+xy} \right|_0^{\frac{1}{y}} = \frac{1}{2} \int_1^{\infty} \frac{dy}{1+y^2} \\ = \frac{1}{2} \lim_{\eta \rightarrow \infty} \int_1^{\eta} \frac{d\eta}{1+\eta^2} = \frac{1}{2} \lim_{\eta \rightarrow \infty} \left| \tan^{-1} \eta \right|_1^{\eta} \\ = \frac{1}{2} \lim_{\eta \rightarrow \infty} [\tan^{-1} \eta - \tan^{-1} 1] = \frac{1}{2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{8}. \\ \therefore U = U_1 + U_2 = \frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{8} = \frac{\pi-1}{4}.$$

5. Establish Dirichlet's formula :

$$\int_0^a dx \int_0^x f(x, y) dy = \int_0^a dy \int_y^a f(x, y) dx.$$

[This follows at once from Example 1.]

6. Prove that

$$\int_0^x \left(\int_0^u f(t) dt \right) du = \int_0^x f(u) (x-u) du,$$

where $f(x)$ is a continuous function.

(C. H. 1965)

[Since $f(t)$ is a continuous function of t , the double integral

$$\int_R \int f(t) du dt \text{ exists and is equal to } \int_0^x du \int_0^u f(t) dt.$$

Let OA and OB be two perpendicular lines ; taking OA as the u -axis and OB as the t -axis, let N be a point on OA such that $ON=x$. Let the line $t=u$ meet $u=x$ at P . Then the triangle OPN is the field of integration R of the double integral, since u varies from 0 to x , and for any u , t varies from 0 to u . Changing the order, we find that as t varies from 0 to x , for any t , u varies from t to x ;

$$\begin{aligned} \therefore \int_0^x du \int_0^u f(t) dt &= \int_0^x dt \int_t^x f(t) du = \int_0^x f(t) dt \int_t^x du \\ &= \int_0^x (x-t) f(t) dt = \int_0^x (x-u) f(u) du, \end{aligned}$$

by changing the variable from t to u , as such a change does not affect a definite integral.]

7. By changing the order of integration, evaluate the integral

$$\int_0^1 dx \int_x^{\frac{1}{x}} \frac{y^2 dy}{(x+y)^2 \sqrt{1+y^2}}.$$

$$\left[\sqrt{2} - \frac{1}{2} \right]$$

8. By calculating the double integral

$$\int_0^{\infty} \int_0^{\infty} e^{-xy} \sin ax \, dy \, dx$$

in two different ways, show that

$$\int_0^{\infty} \frac{\sin ax}{x} dx = \pm \frac{\pi}{2},$$

provided that a is not zero. (Goursat).

$$\begin{aligned} \left[\int_0^{\infty} e^{-xy} dy \right]_{Y \rightarrow \infty}^Y &= \int_0^Y e^{-xy} dy = \left[\frac{e^{-xy}}{-x} \right]_0^Y \\ &= \left[\frac{1}{x} - \frac{1}{xY} \right]_{Y \rightarrow \infty} = \frac{1}{x}. \end{aligned}$$

$$\therefore U = \int_0^{\infty} \int_0^{\infty} e^{-xy} \sin ax \, dy \, dx = \int_0^{\infty} \frac{\sin ax}{x} dx. \quad \dots \quad (1)$$

$$\begin{aligned} \text{Again } \int_0^{\infty} e^{-xy} \sin ax \, dx &= \left[\frac{e^{-xy} (-y \sin ax - a \cos ax)}{a^2 + y^2} \right]_0^X \\ &= \left[\frac{e^{-Xy} (-y \sin aX - a \cos aX)}{a^2 + y^2} + \frac{a}{a^2 + y^2} \right]_{Y \rightarrow \infty} = \frac{a}{a^2 + y^2}. \end{aligned}$$

$$\begin{aligned} \therefore U &= \int_0^{\infty} \int_0^{\infty} e^{-xy} \sin ax \, dx \, dy = \int_0^{\infty} \frac{a \, dy}{a^2 + y^2} \\ &= \left[\frac{a \, dy}{a^2 + y^2} \right]_{Y \rightarrow \infty}^Y = \left[\tan^{-1} \frac{y}{a} \right]_0^Y \\ &= \left[\tan^{-1} \frac{Y}{a} \right]_{Y \rightarrow \infty} = \pm \frac{\pi}{2}, \text{ according as } a > 0 \text{ or } < 0. \quad \dots (2) \end{aligned}$$

From (1) and (2)

$$\int_0^{\infty} \frac{\sin ax}{x} dx = \pm \frac{\pi}{2} \text{ according as } a > 0 \text{ or } < 0.]$$

CHAPTER 21

INFINITE SERIES & INFINITE PRODUCTS

21.1. Definitions.

Let u_n be a function of n which has a definite value for all positive integral values of n . An expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

in which every term is followed by another term is called an *infinite series*.

This series will be denoted by $\sum_1^{\infty} u_n$ or simply Σu_n , and the sum of its first n terms by S_n . S_n is sometimes called the n th partial sum.

As n tends to infinity, there are four distinct possibilities: S_n may tend to a definite finite limit, to plus infinity, to minus infinity or it may do none of these things.

If S_n tends to a definite limit S , the series is said to be *convergent* and S is called its *sum to infinity* or simply its *sum*. Thus S is defined by $\text{Lt } S_n = S$. This is also expressed by writing $u_1 + u_2 + u_3 + \cdots$ to $\infty = S$, or $\Sigma u_n = S$.

If S_n tends to $+\infty$ or to $-\infty$, the series is said to be *divergent*.

If S_n does not tend to a limit, finite or infinite, the series is said to *oscillate*. The series oscillates finitely or infinitely according as S_n oscillates between finite limits or between $+\infty$ and $-\infty$.

Illustrations :

- (i) Σu_n where $u_n = \frac{1}{n(n+1)}$ is convergent and its sum is 1.
- (ii) Σu_n where $u_n = n$ is divergent and $S_n \rightarrow \infty$.
- (iii) Σu_n where $u_n = -n$ is divergent and $S_n \rightarrow -\infty$.
- (iv) (a) Σu_n where $u_n = (-1)^{n+1}$ oscillates between 1 and 0.
 (b) Σu_n where $u_n = (-1)^{n+1} \cdot n$ oscillates between $-\infty$ and $+\infty$.

Note 1. If $u_1, u_2, u_3, \dots, u_n, \dots$ are constants, real or complex, $\sum u_n$ is a series of constant terms. If however $u_1, u_2, u_3, \dots, u_n, \dots$ are variables, say functions of a variable x , $\sum u_n$ is an infinite series whose terms are functions of x . A special type of this kind of series is a 'power series' which has been discussed in Chapter 14.

Note 2. $S_1, S_2, S_3, \dots, S_n, \dots$ form an infinite sequence. If the sequence $\{S_n\}$ converges to S , $\sum u_n$ is convergent and its sum is S . If $\{S_n\}$ diverges or oscillates, $\sum u_n$ also diverges or oscillates.

Note 3 The sum to infinity of an infinite series is not a 'sum' in the ordinary sense of the word, because the word sum means the result obtained from a process of successive addition which eventually terminates. The operation of adding up the terms of an infinite series can never be completed. But if a series is convergent, the successive sums tend to a limit, and the sum to infinity is not a 'sum' but 'the limit of a sum'. The phrase 'sum to infinity' is restricted to convergent series. We are therefore not justified in assuming that the sum of an infinite series is not altered by changing the order of the terms or by introduction or removal of brackets. In fact, changes of this kind may alter the sum, or may transform a convergent series into one which diverges or oscillates. Thus the series $(1-1) + (1-1) + (1-1) + \dots$ is convergent and its sum is zero; but the series $1-1+1-1+1-1+\dots$ oscillates.

Note 4. Rules of algebra cannot always be applied to infinite series. For example, let $S = 1 + 2 + 4 + 8 + 16 + \dots$, then $2S = 2 + 4 + 8 + 16 + \dots$, whence $2S - S = -1$, i.e., $S = -1$ which is absurd, as all the terms in S are positive. If however a series is convergent, it is equal to a finite quantity and we can handle it as such.

21.2. The Geometric Series :

The series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$, ($a > 0$) ... (1)
is an infinite geometric progression with common ratio r .

(i) If $|r| < 1$, it is convergent and its sum is $\frac{a}{1-r}$.

For $S_n = \frac{a(1-r^n)}{1-r}$ and as $n \rightarrow \infty$, $r^n \rightarrow 0$, and so $S_n \rightarrow \frac{a}{1-r}$.

(ii) If $r = 1$, the series is $a + a + a + \dots + a + \dots$... (2)

Here $S_n = na$ and however small the fixed number a may be, $na \rightarrow \infty$ as $n \rightarrow \infty$. The series in this case is divergent.

(iii) If $r > 1$, any term of the series (1) is greater than the

corresponding term of series (2); hence for $r > 1$, the series is *a fortiori* divergent.

(iv) If $r = -1$, the series is $a - a + a + a + \dots$ which oscillates between a and 0 .

(v) If $r < -1$, the terms of (1) numerically increase beyond limit. Hence the series oscillates between $+\infty$ and $-\infty$.

21.3. Theorems : (I). If m is a given positive integer, the series $u_1 + u_2 + u_3 + \dots$ and $u_{m+1} + u_{m+2} + u_{m+3} + \dots$ both converge, both diverge or both oscillate.

For let $S_n = u_1 + u_2 + u_3 + \dots + u_n$,

and $\sigma_n = u_{m+1} + u_{m+2} + u_{m+3} + \dots + u_{m+n}$;

then $\sigma_n = S_{m+n} - S_m$ and S_m is a fixed number; hence if (i) S_{m+n} tends to a finite limit or to $\pm\infty$, so will σ_n ; (ii) S_{m+n} tends to no limit, whether finite or infinite, neither will σ_n .

Cor. The character of a series as regards convergency is not affected by adding, removing or altering a finite number of terms at the beginning.

(II). If k is a fixed number, and Σu_n converges to a sum S , then Σku_n is convergent and its sum is kS . Also if Σu_n diverges or oscillates, so does Σku_n , unless $k=0$.

For if $S_n \rightarrow S$, then $kS_n \rightarrow kS$. If $k \neq 0$ and $S_n \rightarrow \pm\infty$, so does kS_n . If S_n tends to no fixed limit, neither does kS_n .

21.4. Addition and subtraction of convergent series.

If the two series Σu_n , and Σv_n both converge and their sums are s and σ respectively, then (i) $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) + \dots$, converges and its sum is $s + \sigma$; (ii) $(u_1 - v_1) + (u_2 - v_2) + (u_3 - v_3) + \dots$ converges and its sum is $s - \sigma$.

(i) For $\text{Lt} \{(u_1 + v_1) + (u_2 + v_2) + \dots + (u_n + v_n)\}$
 $= \text{Lt} (u_1 + u_2 + \dots + u_n) + \text{Lt} (v_1 + v_2 + \dots + v_n)$
 $= s + \sigma.$

(ii) The proof is similar to the preceding.

Note. In the proof, the series in the brackets are all finite.

Cor. 1. If Σu_n and Σv_n are both convergent with sums s and σ respectively, $\Sigma (au_n \pm bv_n)$ is convergent, with $as \pm b\sigma$ as sums, a, b being constants.

Cor. 2. If Σu_n is convergent then $\text{Lt } u_n$ is necessarily equal to zero.

21'5 Positive series.

A series in which all the terms are not negative is a series of positive terms or simply a positive series. This includes series in which all the terms are positive beginning with a particular term. We shall now consider such series.

21'6. Theorem.

A positive series cannot oscillate; it converges if a number k exists such that $S_n \leq k$ for all values of n and its sum is less than or equal to k ; if no such number k exists, it diverges to $+\infty$.

The sum-sequence $\{S_n\}$ is a monotone increasing sequence; if a number k exists such that $S_n \leq k$, the sequence is bounded above and so S_n tends to a limit $\leq k$; the corresponding infinite series then converges to a sum less than or equal to k . If no such k exists, the sequence is not bounded and so S_n tends to infinity; the corresponding infinite series then diverges to $+\infty$.

21'7. Theorem. The convergency or divergency of a positive series is not affected by the introduction or removal of brackets, the order of the terms remaining unaltered.

Let Σv_n be the series obtained by the grouping together in brackets of the terms of Σu_n and let σ_n and S_n be the n -th partial sums of Σv_n and Σu_n . If u_p be the last term in v_n , $\sigma_n = S_p$, also p increases with n and both tend to infinity simultaneously. Therefore $\text{Lt } \sigma_n = \text{Lt } S_p$. If $\text{Lt } S_p = S$, then $\text{Lt } \sigma_n = S$ and conversely. If $S_p \rightarrow \infty$, so does σ_n and conversely.

21'8. Dirichlet's theorem on changing the order of terms.

If the terms of a convergent positive series be rearranged, the series remains convergent and its sum is unaltered.

Let Σu_n be convergent, and let the terms be re-arranged in any manner. Denote the new series by Σv_n , so that every u is a v and every v is a u . Let S be the sum of Σu_n . Then the sum of any number of terms, selected from the u 's, is not greater than S . But every v is a u and therefore the sum of any number of terms selected from the v 's is not greater than S . Hence Σv_n is convergent and its sum σ is not greater than S , i.e., $\sigma \leq S$.

It can be shown in exactly the same way that $S \leq \sigma$. Thus $\sigma = S$.

The argument fails for such a derangement as

$$u_1 + u_3 + u_5 + \dots + u_2 + u_4 + u_6 + \dots,$$

where $\sum u_n$ is broken up into two (or any finite number of) infinite series. The theorem was first explicitly stated by Dirichlet in 1837, but was no doubt known to earlier writers and in particular to Cauchy.

21'9. An important series.

The series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

Since $\sum \frac{1}{n^p}$ is a positive series, its convergency or divergency is not affected by grouping the terms in brackets.

First let $p > 1$. On grouping the terms as follows :

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p} \right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right) + \dots, \quad \dots \quad (1)$$

where the first terms in the brackets are $\frac{1}{2^p}$, $\frac{1}{2^{2p}}$, $\frac{1}{2^{3p}}$, etc. and the brackets contain $2, 2^2, 2^3, \dots$ terms respectively, we find that each term in (1) after the first is less than the corresponding term in

$$1 + \left(\frac{1}{2^p} + \frac{1}{2^p} \right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} \right) + \dots,$$

which is the same as

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \dots \text{ or, } 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \dots \quad (2)$$

Now (2) is a geometric series whose common ratio $\frac{1}{2^{p-1}}$ is less than unity as $p > 1$. Hence (2) is convergent and therefore (1), i.e., $\sum \frac{1}{n^p}$ is convergent.

Next let $p = 1$. The series now is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

This is sometimes called the *harmonic series*. Group the terms thus :

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \quad (3)$$

where the last terms in the brackets are $\frac{1}{2^2}$, $\frac{1}{2^3}$, $\frac{1}{2^4}$, etc., and the brackets contain 2, 2^2 , 2^3 , ... terms respectively. Each term of (3) is greater than the corresponding term of

$$1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots, \text{ or, } 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \quad (4)$$

Now (4) is obviously divergent and so $\sum \frac{1}{n}$ is divergent.

Lastly if $p < 1$, each term of $\sum \frac{1}{n^p}$ after the first is greater than the corresponding term of $\sum \frac{1}{n}$ and $\sum \frac{1}{n^p}$ in this case is divergent.

21.10. D' Alembert's Ratio Test.

A positive series $\sum u_n$ is convergent if $\frac{u_{n+1}}{u_n} < k < 1$, where k is a fixed number and n may have any value. The series is divergent if $\frac{u_{n+1}}{u_n} \geq 1$ for all values of n .

For if $\frac{u_{n+1}}{u_n} < k < 1$ for every n , then

$$\frac{u_n}{u_1} = \frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \dots \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} < k^{n-1}.$$

Thus $u_n < u_1 k^{n-1}$ and $\sum u_1 k^{n-1}$ is convergent as $k < 1$; therefore $\sum u_n$ is convergent.

Again, if $\frac{u_{n+1}}{u_n} \geq 1$ for all values of n , then $u_n \geq u_{n-1} \geq u_{n-2} \geq \dots \geq u_1$; therefore $u_1 + u_2 + \dots + u_n \geq nu_1$ and nu_1 tends to infinity with n . Hence $\sum u_n$ is divergent.

In particular, if $\frac{u_{n+1}}{u_n}$ tends to a limit p , then $\sum u_n$ is convergent when $p < 1$ and divergent when $p > 1$.

For since $\text{Lt } \frac{u_{n+1}}{u_n} = p$, corresponding to any arbitrary positive ϵ , we can find a positive integer m such that

$$p - \epsilon < \frac{u_{n+1}}{u_n} < p + \epsilon \text{ when } n > m.$$

Now if $p < 1$, we can choose ϵ sufficiently small to make $p + \epsilon = k < 1$, and by what has been proved above, $u_{m+1} + u_{m+2} + u_{m+3} + \dots$ is convergent. It follows that Σu_n is convergent.

Again if $p > 1$, we can choose ϵ to make $p - \epsilon = k > 1$ for $n > m$.
 $\therefore u_{m+1} + u_{m+2} + u_{m+3} + \dots$ is divergent and consequently Σu_n is divergent.

Note. It is sufficient that the conditions hold for all values of n greater than some fixed value. Again nothing is said about the case in which $\text{Lt } \frac{u_{n+1}}{u_n} = 1$. In this case, the series may converge or diverge and we say that the test fails.

21.11. Cauchy's Root Test.

A positive series Σu_n is convergent if $u_n^{\frac{1}{n}} < k < 1$ where k is a fixed number and n has any value. The series is divergent if $u_n^{\frac{1}{n}} \geq 1$ for infinitely many values of n .

If $u_n^{\frac{1}{n}} < k < 1$, then $u_n < k^n$ and since $k < 1$, the series Σk^n is convergent and consequently Σu_n is convergent.

If $u_n^{\frac{1}{n}} \geq 1$ for infinitely many values of n , we have $u_n \geq 1$ for such values of n . Σu_n is then obviously divergent.

In particular, if $u_n^{\frac{1}{n}}$ tends to a limit p , then Σu_n is convergent if $p < 1$ and divergent if $p > 1$. The proof is similar to that of the last theorem.

It is sufficient that the conditions hold for $n > m$, where m is a fixed number. Also nothing is said about the case in which $\text{Lt } u_n^{\frac{1}{n}} = 1$; if however, the limit is approached from above, Σu_n is divergent.

Note. In Art. 4.14 we have seen that if $\frac{u_{n+1}}{u_n} \rightarrow p$, then also $u_n^{\frac{1}{n}} \rightarrow p$; but the converse is not true, that is, if $u_n^{\frac{1}{n}}$ tends to a limit, $\frac{u_{n+1}}{u_n}$ may not tend to any limit. It follows that Cauchy's test can be applied whenever D'Alembert's test is applicable and sometimes where the latter fails. Thus if $0 < a < b < 1$, the series $a + b + a^2 + b^2 + a^3 + b^3 + \dots$ is convergent. Here $u_n^{\frac{1}{n}} = a^{\frac{1}{2}}$ or $b^{\frac{1}{2}}$ according as n is odd or even and as both a and b are less than 1, the series is convergent. Now $\frac{u_{2n}}{u_{2n-1}} = \left(\frac{b}{a}\right)^n$ and $\frac{u_{2n+1}}{u_{2n}} = a \left(\frac{a}{b}\right)^n$. Thus $\text{Lt} \frac{u_{n+1}}{u_n}$ does not exist and D'Alembert's test fails.

Cauchy's test may be said to be more general than D'Alembert's test; but the latter test is more useful because in most series $\frac{u_{n+1}}{u_n}$ is a simpler function of n than u_n .

21.12. Cauchy's Condensation Test.

If $f(n)$ is a positive decreasing function of n and a is a positive integer greater than 1, then the two series $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ are both convergent or both divergent.

Group the terms of $\Sigma f(n)$ as follows :

$$\begin{aligned} & \{f(1) + f(2) + \dots + f(a)\} + \{f(a+1) + f(a+2) + \dots + f(a^2)\} \\ & \quad + \{f(a^2+1) + f(a^2+2) + \dots + f(a^3)\} + \dots + \{f(a^{n-1}+1) \\ & \quad + f(a^{n-1}+2) + \dots + f(a^n)\} + \dots, \end{aligned}$$

where if v_n denotes the sum of the terms of the n -th group,

$$v_n = f(a^{n-1}+1) + f(a^{n-1}+2) + \dots + f(a^n).$$

The number of terms in v_n is $a^n - a^{n-1}$ and since $f(n)$ is a decreasing function.

$$(a^n - a^{n-1})f(a^n) \leq v_n \leq (a^n - a^{n-1})f(a^{n-1});$$

$$\text{or, } \frac{1}{a}(a-1)a^n f(a^n) \leq v_n \leq (a-1)a^{n-1}f(a^{n-1}),$$

$$\therefore \frac{1}{a}(a-1)\Sigma a^n f(a^n) \leq \Sigma v_n = \Sigma f(n), \dots \dots (1)$$

and Σv_n or $\Sigma f(n) \leq (a-1)\{f(1) + \Sigma a^n f(a^n)\} \dots \dots (2)$

From (1) it follows that if $\Sigma f(n)$ is convergent, so also is $\Sigma a^n f(a^n)$ and if $\Sigma a^n f(a^n)$ is divergent, $\Sigma f(n)$ is also divergent. Again from (2), we find that if $\Sigma f(n)$ is divergent, so is $\Sigma a^n f(a^n)$ and if $\Sigma a^n f(a^n)$ is convergent, so is $\Sigma f(n)$.

[The theorem has been proved to be true for values, not necessarily integral, of a greater than 1.]

21.13. An Important Series.

The series $\Sigma \frac{1}{n(\log n)^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

If a is any positive integer other than unity, the series is convergent or divergent according as the series Σv_n is convergent or divergent where

$$v_n = \frac{a^n}{a^n (\log a^n)^p} = \frac{1}{(\log a)^p} \cdot \frac{1}{n^p}. \quad (\text{Art. 21.12})$$

Since $(\log a)^p$ is a constant, the given series is convergent, or divergent according as $\Sigma \frac{1}{n^p}$ is convergent or divergent, that is, according as $p > 1$ or $p \leq 1$.

21.14. A comparison test.

1. If Σu_n and Σv_n be two positive series, of which Σv_n is convergent, then Σu_n is convergent if either

(i) $u_n \leq cv_n$ for all values of n greater than a fixed integer m ;

or, (ii) $\text{Lt} \frac{u_n}{v_n} = p$,

where c and p are positive constants.

Proof: (i) If σ be the sum to infinity of Σv_n , we have

$$v_1 + v_2 + v_3 + \dots + v_n < \sigma$$

for all values of n . Therefore

$$u_{m+1} + u_{m+2} + \dots + u_n \leq c(v_{m+1} + v_{m+2} + \dots + v_n) < c\sigma$$

for all values of n greater than m . But m is fixed, therefore Σu_n is convergent.

(ii) Since $\text{Lt } \frac{u_n}{v_n} = p$, for any positive ϵ , we can find such that

$$\frac{u_n}{v_n} < p + \epsilon \text{ for } n > m.$$

Hence it follows from (i) that Σu_n is convergent.

Note. p may be 0 in (ii).

2. If Σu_n and Σv_n be two positive series of which Σv_n is divergent then Σu_n is divergent if either

(i) $u_n \geq cv_n$ for all values of n greater than a fixed integer m ;
or, (ii) $\text{Lt } \frac{u_n}{v_n} = p$,

where c and p are positive constants.

Proof: (i) Since Σv_n is divergent and m is fixed, $v_{m+1} + v_{m+2} + \dots + v_n$ takes values as large as we please for all sufficiently large values of n . But $u_{m+1} + u_{m+2} + \dots + u_n \geq c(v_{m+1} + v_{m+2} + \dots + v_n)$ and $c > 0$; therefore $u_{m+1} + u_{m+2} + \dots + u_n$ takes values as large as we please for sufficiently large values of n ; and hence Σu_n is divergent.

(ii) Since $\text{Lt } \frac{u_n}{v_n} = p$, corresponding to any positive ϵ , we can find m such that

$$p - \epsilon < \frac{u_n}{v_n} < p + \epsilon$$

for $n > m$. Also ϵ may be chosen sufficiently small to make $p - \epsilon = c > 0$;

$$\therefore u_n > (p - \epsilon)v_n > cv_n ;$$

and so by (i), Σu_n is divergent.

Note. In (ii), p may be ∞ .

21.15. A second comparison test.

Let Σu_n and Σv_n be two positive series ; then

(i) Σu_n is convergent when Σv_n is convergent

$$\text{if } \frac{u_n}{v_n} > \frac{v_n}{v_{n+1}} \text{ for } n \geq m ;$$

(ii) Σu_n is divergent when Σv_n is divergent

$$\text{if } \frac{u_n}{v_n} < \frac{v_n}{v_{n+1}} \text{ for } n \geq m.$$

Proof: (i) Rejecting the first $m-1$ terms for which the condition does not hold and writing the series from the m th term as $u_1+u_2+u_3+\dots$, and $v_1+v_2+v_3+\dots$, we have, since

$$\frac{u_1}{u_2} > \frac{v_1}{v_2}, \frac{u_2}{u_3} > \frac{v_2}{v_3}, \frac{u_3}{u_4} > \frac{v_3}{v_4}, \dots,$$

$$\begin{aligned} \text{Now } u_1+u_2+u_3+\dots &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \dots \right) \\ &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \dots \right) \\ &< u_1 \left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} \cdot \frac{v_2}{v_1} + \dots \right) \\ &< \frac{u_1}{v_1} (v_1+v_2+v_3+\dots). \end{aligned}$$

Hence if the series Σv_n is convergent, so is Σu_n .

$$\begin{aligned} \text{(ii) Again } u_1+u_2+u_3+\dots &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \dots \right), \\ &> u_1 \left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} \cdot \frac{v_2}{v_1} + \dots \right), \\ &> \frac{u_1}{v_1} (v_1+v_2+v_3+\dots). \end{aligned}$$

It follows that if Σv_n is divergent, so is Σu_n .

This test may be stated in the form :

If Σv_n be convergent and $\text{Lt } \frac{u_n}{u_{n+1}} > \text{Lt } \frac{v_n}{v_{n+1}}$, then Σu_n is convergent; and if Σv_n be divergent and $\text{Lt } \frac{u_n}{u_{n+1}} < \text{Lt } \frac{v_n}{v_{n+1}}$, then Σu_n is divergent.

21'16. Raabe's Test (or, Duhamel's test).

Σu_n , a positive series, is convergent or divergent according as

$$\text{Lt } \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > 1, \text{ or } < 1;$$

if the limit is equal to 1, the test fails.

Let us compare Σu_n with the series whose general term is $\frac{1}{n^p}$.

When $p > 1$, the series $\sum \frac{1}{n^p}$ is convergent and in this case $\sum u_n$ is convergent if

$$\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)^p, \text{ i. e., } > \left(1 + \frac{1}{n}\right)^p;$$

$$\text{or, } \frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \dots,$$

$$\text{or, } n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2n} + \dots,$$

$$\text{that is } \text{Lt} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} \geq p > 1.$$

The other case can be similarly proved.

21.17. The logarithmic form of Raabe's test.

A positive series $\sum u_n$ is convergent or divergent according as

$$\text{Lt} \left(n \log \frac{u_n}{u_{n+1}} \right) > 1, \text{ or } < 1;$$

the test fails if the limit is equal to 1.

As above, if $p > 1$, $\sum u_n$ is convergent if

$$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p,$$

$$\text{i.e., if } \log \frac{u_n}{u_{n+1}} > p \log \left(1 + \frac{1}{n}\right);$$

$$\text{or, if } \log \frac{u_n}{u_{n+1}} > \frac{p}{n} - \frac{p}{2n^2} + \dots;$$

$$\text{i.e., if } \text{Lt} \left(n \log \frac{u_n}{u_{n+1}} \right) \geq p > 1.$$

The second part can be similarly established.

21.18. Bertrand's or De Morgan's test.

A positive series $\sum u_n$ is convergent or divergent according as

$$\text{Lt} \left[\left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right\} \log n \right] > 1 \text{ or } < 1;$$

the test fails if the limit is 1.

Compare the series with Σv_n where $v_n = \frac{1}{n(\log n)^p}$. Σv_n is convergent if $p > 1$ and divergent if $p < 1$ (Art. 21'13).

Hence if $p > 1$, Σu_n is convergent if

$$\begin{aligned} \frac{u_n}{u_{n+1}} &> \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p}, \\ &> \left(1 + \frac{1}{n}\right) \left\{ \frac{\log(n+1)}{\log n} \right\}^p, \\ &> \left(1 + \frac{1}{n}\right) \left\{ \frac{\log n + \log\left(1 + \frac{1}{n}\right)}{\log n} \right\}^p, \\ &> \left(1 + \frac{1}{n}\right) \left\{ \frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \dots}{\log n} \right\}^p, \\ &> \left(1 + \frac{1}{n}\right) \left\{ 1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \dots \right\}^p, \\ &> 1 + \frac{1}{n} + \frac{p}{n \log n} + \dots, \end{aligned}$$

or, $n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 > \frac{p}{\log n} + \text{terms containing } n \text{ and } \log n \text{ in the denominator,}$

or, if $\text{Lt} \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] \geq p > 1$.

$\therefore$ if $\text{Lt} \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = \lambda$,

Σu_n is convergent if $\lambda > 1$.

The case where $p < 1$ can be similarly dealt with.

21'19. Kummer's test (as modified by Dini).

Let Σu_n and $\sum \frac{1}{d_n}$ be two positive series and suppose $\sum \frac{1}{d_n}$ is divergent; further suppose that

$$v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1};$$

then (i) if a fixed number k can be found so that for $n \geq m$, $v_n > k > 0$, the series Σu_n is convergent;

(ii) If $v_n < 0$ for $n \geq m$, $\sum u_n$ is divergent.

Proof: (i) Putting $m, m+1, \dots, n-1$ for n in the given condition $v_n > k$, we have

$$\begin{aligned} d_m u_m - d_{m+1} u_{m+1} &> k u_{m+1}, \\ d_{m+1} u_{m+1} - d_{m+2} u_{m+2} &> k u_{m+2}, \\ &\dots\dots\dots \\ d_{n-1} u_{n-1} - d_n u_n &> k u_n, \end{aligned}$$

whence

$$u_{m+1} + u_{m+2} + \dots + u_n < \frac{1}{k} (d_m u_m - d_n u_n) < \frac{1}{k} d_m u_m.$$

Thus if S_r be the sum of the first r terms of $\sum u_n$, we have

$$S_n < S_m + \frac{1}{k} d_m u_m, \text{ a fixed number;}$$

therefore $\sum u_n$ is convergent.

(ii) Here since $d_n \frac{u_n}{u_{n+1}} - d_{n+1} < 0$ for $n \geq m$, we have

$$d_m u_m < d_{m+1} u_{m+1} < \dots < d_n u_n;$$

therefore $u_n > \frac{d_m u_m}{d_n}$. Now $d_m u_m$ is a fixed number and $\sum \frac{1}{d_n}$ is divergent; hence $\sum u_n$ is divergent.

Note. If $v_n \rightarrow 0$, we cannot find k so that $v_n > k > 0$ for $n \geq m$, and the test fails. In part (i) of the test, $\sum \frac{1}{d_n}$ need not diverge; in fact d_n may be any positive function of n .

21.20. Deductions from Kummer's test.

(i) If we take $d_n = 1$, we get D'Alembert's test.

(ii) If we take $d_n = n$, we get Raabe's test.

(iii) $d_n = n \log n$ gives Bertrand's test.

21.21. Gauss's test.

If $\sum u_n$ be a positive series and $\frac{u_n}{u_{n+1}}$ can be expressed in the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p},$$

where $p > 1$ and $|b_n| < a$ fixed number k , or in particular, b_n

tends to a finite limit as $n \rightarrow \infty$, then Σu_n is convergent if $a > 1$ and divergent if $a \leq 1$.

Suppose in the first place $a \neq 1$; then

$$n \left(\frac{u_n}{u_{n+1}} - 1 \right) = a + \frac{b_n}{n^{p-1}} \rightarrow a \text{ as } n \rightarrow \infty,$$

for $|b_n| < k$ and $p > 1$. Therefore by Raabe's test Σu_n converges if $a > 1$ and diverges if $a < 1$.

If $a = 1$, Raabe's test fails and we apply Bertrand's. In this case

$$\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n = \frac{b_n \log n}{n^{p-1}},$$

and as $|b_n| < k$ and $\frac{\log n}{n^{p-1}} \rightarrow 0$ as $p > 1$, we have

$$\text{Lt} \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = 0.$$

Hence the series is divergent.

Cor. If $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b}{n^2} + \dots$, the series being convergent for sufficiently large values of n , then Σu_n is convergent if $a > 1$ and divergent if $a \leq 1$. For in this case b_n is the sum of the convergent series.

21.22. Cauchy-Maclaurin Integral test.

Let Σu_n be a positive series such that u_n steadily decreases as n increases and let $f(x)$ be a positive monotone decreasing function of x for $x \geq 1$ such that $f(n) = u_n$ for all positive integral values of n ,

then the series Σu_n and the improper integral $\int_1^{\infty} f(x) dx$ both con-

verge or both diverge.

Since $f(x)$ is monotone decreasing, $f(n-1) \geq f(x) \geq f(n)$ where $n-1 \leq x \leq n$, for all positive integers $n \geq 2$.

$$\therefore \int_{n-1}^n f(n-1) dx \geq \int_{n-1}^n f(x) dx \geq \int_{n-1}^n f(n) dx,$$

$$\text{or, } f(n-1) \geq \int_{n-1}^n f(x) dx \geq f(n),$$

$$\text{i.e., } u_{n-1} \geq \int_{n-1}^n f(x) dx \geq u_n. \quad \dots \quad (1)$$

If n be given successively the values 2, 3, 4, ..., n and the inequalities be added together, we get

$$u_1 + u_2 + \dots + u_{n-1} \geq \int_1^n f(x) dx \geq u_2 + u_3 + \dots + u_n;$$

$$\therefore S_n - u_1 \leq \int_1^n f(x) dx \leq S_n - u_n,$$

$$\text{or, } u_n \leq S_n - \int_1^n f(x) dx \leq u_1. \quad \dots \quad (2)$$

Putting $\phi(n)$ for $S_n - \int_1^n f(x) dx$, we find

$$\begin{aligned} \phi(n) - \phi(n-1) &= S_n - \int_1^n f(x) dx - S_{n-1} + \int_1^{n-1} f(x) dx \\ &= u_n - \int_{n-1}^n f(x) dx \leq 0, \quad [\text{from (1)}]. \end{aligned}$$

$\therefore \phi(n)$ is monotone decreasing and bounded; hence the sequence $\{\phi(n)\}$ is convergent.

$$\therefore \lim_{n \rightarrow \infty} \phi(n) = \lim_{n \rightarrow \infty} \left\{ S_n - \int_1^n f(x) dx \right\} \text{ exists and } = \lambda \text{ say;}$$

$$\text{hence } \lim_{n \rightarrow \infty} \int_1^n f(x) dx = \lim_{n \rightarrow \infty} S_n - \lambda.$$

Therefore if S_n tends to a finite limit, so does $\int_1^{\infty} f(x) dx$ and if

$S_n \rightarrow \infty$, so does $\int_1^{\infty} f(x) dx$. Conversely if $\int_1^{\infty} f(x) dx$ has a finite

value, S_n tends to a finite value and if $\int_1^{\infty} f(x) dx \rightarrow \infty$, so does S_n .

Thus $\sum u_n$ and $\int_1^{\infty} f(x) dx$ converge or diverge together.

Further if $\lim_{n \rightarrow \infty} \int_1^n f(x) dx = \mu$, then since from (2),

$$u_1 - S_n \geq - \int_1^n f(x) dx,$$

$$\text{or, } S_n \leq u_1 + \int_1^n f(x) dx.$$

On making $n \rightarrow \infty$, we get
 $S \leq u_1 + \mu$.

21.23. General Principle of Convergence.

A series $\sum u_n$ converges if the sequence of partial sums $\{S_n\}$ converges. Now a necessary and sufficient condition for the convergency of $\{S_n\}$ is that corresponding to an arbitrary positive ϵ , there should exist a positive integer m such that for every positive integer p

$$|S_{n+p} - S_n| < \epsilon,$$

when $n \geq m$ and $p > 0$, (Art. 4.10).

Now $|S_{n+p} - S_n| = |u_{n+1} + u_{n+2} + \dots + u_{n+p}| = |R_{n,p}|$ say;
 therefore a necessary and sufficient condition for the convergency

of Σu_n is that for an arbitrary positive ϵ , a positive integer m can be found such that

$$|R_{n,p}| = |u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \epsilon$$

for, $n \geq m$ and $p > 0$.

Note, $|R_{n,p}| < \epsilon$ is necessary but not a sufficient condition unless p is allowed to tend to infinity in any manner.

21'24. Series of positive and negative terms.

Introduction and removal of brackets. Let the terms of Σu_n be arranged in groups without altering their order. Denote the sum of the terms in the n th group by v_n ; then

(i) If Σu_n is convergent, so also is Σv_n and the two series have the same sum. For let

$$S_n = u_1 + u_2 + \dots + u_n,$$

$$\sigma_n = v_1 + v_2 + \dots + v_n.$$

Then for any value of n , we can find m so that

$$\text{either } S_n = \sigma_m \text{ or, } S_n = \sigma_m + (\dots + u_n)$$

where the terms in the bracket are included in the group v_{m+1} . If Σu_n be convergent,

$$|R_{n,p}| = |u_{n+1} + u_{n+2} + \dots + u_{n+p}| \rightarrow 0 \text{ as } n \rightarrow \infty;$$

also $m \rightarrow \infty$ as $n \rightarrow \infty$. Now $S_n - \sigma_m = \dots + u_n = u_{k+1} + u_{k+2} + \dots + u_{k+p}$, say, and k increases with m and n . $\therefore |S_n - \sigma_m| = |R_{k,p}| \rightarrow 0$. Hence Σv_n converges to the same sum as Σu_n .

(ii) If Σv_n is convergent, Σu_n is not necessarily so. For given that Σv_n is convergent, we cannot conclude that $|R_{n,p}| \rightarrow 0$ for all values of p .

21'25. Leibnitz's test for alternating series.

An *alternating series* is one in which the terms are alternately positive and negative.

The series $u_1 - u_2 + u_3 - u_4 + \dots$ which is alternating is convergent if the terms steadily decrease in absolute value and tend to the limit zero.

We have

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}). \dots \dots (1)$$

$$\text{Again } S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - u_{2n}. \dots \dots (2)$$

Now as $u_1 > u_2 > u_3 > \dots$, $(u_1 - u_2)$, $(u_2 - u_3)$, $(u_3 - u_4)$, ... are all positive.

From (1), we find S_{2n} is positive and monotone increasing and from (2), we see that $S_{2n} < u_1$.

$$\therefore 0 < S_{2n} < u_1.$$

$$\text{Again } S_{2n+1} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n}) + u_{2n+1}, \quad \dots \quad \dots(3)$$

$$\text{and } S_{2n+1} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n} - u_{2n+1}). \quad \dots(4)$$

From (3) we see that $S_{2n+1} > 0$ and from (4) we notice that S_{2n+1} is monotone decreasing but less than u_1 .

$$\therefore 0 < S_{2n+1} < u_1.$$

Both $\{S_{2n}\}$ and $\{S_{2n+1}\}$ being bounded tend to limits. Further

$$S_{2n+1} = S_{2n} + u_{2n+1};$$

$$\therefore S_{2n+1} - S_{2n} = u_{2n+1} \rightarrow 0.$$

So $\text{Lt } S_{2n+1} = \text{Lt } S_{2n}$ and S_n tends to a unique limit $\leq u_1$.

21'26. (a) Absolutely convergent series.

A series Σu_n is said to be *absolutely convergent* when $\Sigma |u_n|$ converges.

Theorem I. An absolutely convergent series is convergent.

When $\Sigma |u_n|$ is given to be convergent, it does not follow that Σu_n is convergent. We proceed to prove that it is so.

Let Σu_n be absolutely convergent, then by definition $\Sigma |u_n|$ is convergent. Now $u_n + |u_n| = 2u_n$ or 0 according as $u_n > 0$, or $u_n < 0$.

Therefore every term of $\Sigma(u_n + |u_n|)$ is ≥ 0 and less than the corresponding term of the convergent series $\Sigma(|u_n| + u_n)$, i.e., of $2\Sigma |u_n|$. Hence $\Sigma(u_n + |u_n|)$ is convergent, or $\Sigma u_n + \Sigma |u_n|$ is convergent. It follows that Σu_n is convergent.

Or, we may apply the general principle of convergence to prove this.

Since $\Sigma |u_n|$ is convergent, for any positive ϵ , we can find a positive integer m so that

$$|u_{n_1+1}| + |u_{n_1+2}| + \cdots + |u_{n_2}| < \epsilon$$

when $n_2 > n_1 \geq m$. *A fortiori*

$$|u_{n_1+1} + u_{n_1+2} + \cdots + u_{n_2}| < \epsilon$$

and therefore Σu_n is convergent.

Theorem II. If the terms of an absolutely convergent series be rearranged, the series remains convergent and its sum is unaltered.

Denote the rearranged series by Σv_n so that every v is $\{a$ u and *vice versa*.

Let $\Sigma |u_n| = s$ and $\Sigma(u_n + |u_n|) = s'$, then $\Sigma u_n = s' - s$; and since $\Sigma |u_n|$ and $\Sigma(u_n + |u_n|)$ are positive series, their sums are unaltered by rearrangement of terms (Dirichlet's theorem, Art. 21'8); hence $\Sigma |v_n| = s$ and $\Sigma(v_n + |v_n|) = s'$. It follows that $\Sigma v_n = s' - s = \Sigma u_n$. This in an extended form of Dirichlet's theorem.

Theorem III. An absolutely convergent series Σu_n can be expressed as the difference of two convergent positive series.

$$\begin{aligned} \text{Let } v_n &= u_n, \text{ if } u_n > 0, \\ &= 0, \text{ if } u_n < 0; \\ w_n &= 0, \text{ if } u_n > 0, \\ &= -u_n, \text{ if } u_n < 0. \end{aligned}$$

Then v_n and w_n are both positive and so Σv_n and Σw_n are positive series. Now as $u_n = v_n - w_n$ and $|u_n| = |v_n| + |w_n| = v_n + w_n$, we have

$$\Sigma u_n = \Sigma v_n - \Sigma w_n, \text{ and } \Sigma |u_n| = \Sigma v_n + \Sigma w_n.$$

Since Σu_n and $\Sigma |u_n|$ are both convergent, it follows by addition and subtraction that Σv_n and Σw_n are both convergent. Hence $\Sigma u_n = \Sigma v_n - \Sigma w_n$ is the difference of two convergent positive series.

(b) Conditionally convergent or semi-convergent series.

If Σu_n is convergent but $\Sigma |u_n|$ is divergent, Σv_n is called a *conditionally convergent* or a *semi-convergent* series.

Theorem IV. A semi-convergent series is the difference of two divergent positive series.

For, as above, $\Sigma u_n = \Sigma v_n - \Sigma w_n$ and $\Sigma |u_n| = \Sigma v_n + \Sigma w_n$ where Σv_n and Σw_n are two positive series.

If Σv_n and Σw_n be both convergent, $\Sigma |u_n|$ is convergent, which is contrary to hypothesis.

If one of Σv_n and Σw_n be convergent and the other divergent, Σu_n diverges to $+\infty$ or to $-\infty$, which is again contrary to hypothesis.

Hence both Σv_n and Σw_n must be divergent positive series.

We state without proof the following theorem due Riemann.

Theorem V. By a proper derangement of terms, a semi-

convergent series can be made to diverge, oscillate or even converge to any sum whatever.

(This will be illustrated by examples later.)

21.27. Tests of convergence.

A series $\sum u_n$ containing positive as well as negative terms is absolutely convergent if for $n \geq m$,

$$\left| \frac{u_{n+1}}{u_n} \right| < k, \quad \text{or if} \quad \left| u_n^{\frac{1}{n}} \right| < k,$$

where k is a fixed number less than unity.

Note. There are no comparison tests for convergence of conditionally convergent series.

21.28. Product series.

Let $\sum u_n = u_1 + u_2 + u_3 + \dots$ and $\sum v_n = v_1 + v_2 + v_3 + \dots$ be two infinite series. If the product of every term of the first series with every term of the second series be formed and the sum taken, the series thus obtained is called the product series. Let

$$\left. \begin{aligned} \sum w_n = & u_1 v_1 + u_1 v_2 + u_1 v_3 + \dots \\ & + u_2 v_1 + u_2 v_2 + u_2 v_3 + \dots \\ & + u_3 v_1 + u_3 v_2 + u_3 v_3 + \dots \\ & + \dots \end{aligned} \right\} \quad (P)$$

be this series. It would appear from the above that it is the sum of an infinite number of infinite series. But it can be exhibited as a simple infinite series in various ways. The following are two such ways.

$$u_1 v_1 + (u_1 v_2 + u_2 v_1) + (u_1 v_3 + u_2 v_2 + u_3 v_1) + \dots \quad (A)$$

$$u_1 v_1 + (u_1 v_2 + u_2 v_1) + (u_1 v_3 + u_2 v_2 + u_3 v_1) + \dots \quad (B)$$

21.29. Theorem I. If the two series $s = \sum u_n$ and $t = \sum v_n$ be absolutely convergent, the series $\sum w_n = \sum u_p v_q$, formed by taking the sum of the products of a term of the first series with a term of the second series in any order whatever, is absolutely convergent and has the sum st . (Cauchy).

We first prove that the product series is absolutely convergent. Let $\sum \alpha_p \beta_q$ be the series obtained from $\sum u_p v_q$ by replacing u_n by

its modulus α_n and v_n by its modulus β_n . In the sum of the first n terms of $\Sigma \alpha_p \beta_q$, let μ be the greatest among the suffixes of α and β which appear in these n terms. We shall have

$$\sum_1^n \alpha_p \beta_q < \sum_1^\mu \alpha_p \sum_1^\mu \beta_q,$$

as the right side contains all the terms which occur on the left side together with some more positive terms.

Now the factors on the right side remain finite whatever μ may be, so that the sum on the left side is bounded and as it increases with n , it converges. Hence the series $\Sigma u_p v_q$ is absolutely convergent.

Now Σu_n and Σv_n being absolutely convergent, the terms may be written in any order without affecting the sum. If then the product series be formed as in scheme (P) above and arranged in the form (A) and S_n the sum of the first n terms of (A),

$$\begin{aligned} S_n &= \text{the sum of the first } n \text{ terms of (A),} \\ &= \text{the sum of the first } n \text{ terms of the first } n \text{ rows of (P)} \\ &= s_n t_n. \end{aligned}$$

$$\therefore \text{Lt } S_n = \text{Lt } (s_n t_n) = \text{Lt } s_n \cdot \text{Lt } t_n = st.$$

Hence $\Sigma u_p v_q$ converges to the sum st .

Theorem II. If $s = \Sigma u_n$ and $t = \Sigma v_n$ be positive series, the series $\Sigma u_p v_q$, taken in any order, is absolutely convergent and has the sum st .

This follows as a corollary to Theorem I by taking u_p, v_q for α_p, β_q . As a matter of fact, the series $\Sigma u_p v_q$ is now a positive series.

We give below two more theorems without proof on the multiplication of two series.

Theorem III. If $s = \Sigma u_n$ and $t = \Sigma v_n$ be two convergent series, of which one at least is absolutely convergent, then $\Sigma u_p v_q$ is convergent and has the sum st (Mertens).

Theorem IV. If $s = \Sigma u_n$ and $t = \Sigma v_n$ be convergent and the product series $\Sigma u_p v_q$ is also convergent, then it has the sum st . (Abel).

Note. The convergence of power series is discussed in Chapter 14.

21'30. Infinite Products. Definitions.

If $a_1, a_2, \dots, a_n, \dots$ is a sequence of numbers, the product $a_1 a_2 \dots a_n$ will be denoted by P_n or by $\prod_{r=1}^n a_r$. Thus we write

$$P_n = \prod_{r=1}^n a_r = a_1 a_2 \dots a_n.$$

If P_n tends to a finite limit P , different from zero, as $n \rightarrow \infty$, except when a factor of P_n is zero, we say that the infinite product $a_1 a_2 a_3 \dots$ is convergent and that its value is P , or that it converges to P , and we write

$$P = a_1 a_2 a_3 \dots \text{ to } \infty = \prod_{n=1}^{\infty} a_n, \text{ or simply } \prod a_n.$$

If $P_n \rightarrow \infty$ or $-\infty$, or if $P_n \rightarrow 0$ when no factor of P_n is zero, the infinite product is said to diverge or to be divergent.

If P_n oscillates, the infinite product is said to oscillate.

If $a_1, a_2, a_3, \dots$ are real and positive and $P_n = a_1 a_2 \dots a_n$, then $\log P_n = \log a_1 + \log a_2 + \dots + \log a_n$ and $P_n \rightarrow$ a finite limit, to ∞ , to zero or oscillates according as $\log P_n$ tends to a finite limit, to ∞ , to $-\infty$ or oscillates. Thus P_n and $\sum \log a_n$ both converge, both diverge or both oscillate. Now when $P_n \rightarrow 0$, $\sum \log a_n$ diverges to $-\infty$. We say then that the infinite product *diverges* to zero.

If $P_n \rightarrow$ a finite limit, then $P_{n-1} \rightarrow$ the same limit, and if this limit is not zero, $a_n = P_n / P_{n-1} \rightarrow 1$. Thus if $\prod a_n$ is convergent, then $a_n \rightarrow 1$.

It is generally convenient to write an infinite product in the form

$$P = (1+u_1)(1+u_2) \dots (1+u_n) \dots$$

A necessary but not sufficient condition for convergency is that $u_n \rightarrow 0$ and so after a certain stage $|u_n| < 1$.

The character of P as regards convergency will not be affected by removing any finite number of factors from the beginning of the product; consequently there will be no loss of generality in supposing that $|u_n| < 1$ for every n .

The simplest and most important case is when all the factors have the same sign,

21'31. Theorem. If $u_1, u_2, \dots$ are positive and each less than unity, each of the infinite products

$$(1+u_1)(1+u_2)(1+u_3)\cdots$$

$$(1-u_1)(1-u_2)(1-u_3)\cdots$$

is convergent or divergent according as the series Σu_n is convergent or divergent.

Suppose that Σu_n is convergent and that its sum is s . Then m can be found so that

$$u_{m+1} + u_{m+2} + \cdots \text{to } \infty < 1.$$

For since $u_1 + u_2 + \cdots \text{to } \infty = s$,

$$u_{m+1} + u_{m+2} + \cdots \text{to } \infty = s - s_m.$$

Now $\text{Lt } (s - s_m) = s - s = 0$. $\therefore |s - s_n| < \epsilon$ for $n \geq m$.

i.e., $u_{m+1} + u_{m+2} + \cdots \text{to } \infty < \epsilon < 1$.

Also the omission of the first m factors of an infinite product does not affect its convergency. Hence there is no loss of generality in supposing $s < 1$ and consequently $s_n < s < 1$, for every n . Now we have

$$P_n = (1+u_1)(1+u_2)\cdots(1+u_n) < \frac{1}{1-s_n} < \frac{1}{1-s},$$

$$Q_n = (1-u_1)(1-u_2)\cdots(1-u_n) > 1-s_n > 1-s.$$

Also P_n increases and Q_n decreases as n increases and both are bounded, one above and the other below. So P_n and Q_n tend to positive limits as $n \rightarrow \infty$, and both products $\prod (1+u_n)$ and $\prod (1-u_n)$ are convergent.

Next suppose that Σu_n is divergent. Then

$$P_n = (1+u_1)(1+u_2)\cdots(1+u_n) > 1+s_n \rightarrow \infty,$$

$$Q_n = (1-u_1)(1-u_2)\cdots(1-u_n) < \frac{1}{1+s_n} \rightarrow 0.$$

Moreover $Q_n > 0$. Therefore $P_n \rightarrow \infty$ and $Q_n \rightarrow 0$; so that $\prod (1+u_n)$ diverges to ∞ and $\prod (1-u_n)$ diverges to 0.

Note. We have used the inequalities of Weierstrass in the above discussion.

21'32. A necessary and sufficient condition for convergency.

A necessary and sufficient condition for the convergency of an infinite product $P = a_1 a_2 a_3 \dots$ is :

Corresponding to any arbitrary positive ϵ , there is an integer m such that for every positive integer p

$$|a_{m+1}a_{m+2}\dots a_{m+p}-1| < \epsilon$$

$$\text{or, } \left| \frac{P_{m+p}}{P_m} - 1 \right| < \epsilon. \quad \dots \quad (1)$$

Let P be convergent. Then P_n tends to a limit different from zero, and so a positive number λ exists such that

$$|P_n| > \lambda$$

for every n .

Now by the general principle of convergence applied to the sequence $\{P_n\}$, we can find m such that

$$|P_{m+p}-P_m| < \lambda\epsilon$$

for every positive integer p .

$$\therefore \left| \frac{P_{m+p}}{P_m} - 1 \right| < \frac{\lambda\epsilon}{|P_m|} < \frac{\lambda\epsilon}{\lambda} = \epsilon,$$

so that the condition (1) holds.

Conversely if the condition (1) holds then

$$1-\epsilon < \frac{P_{m+p}}{P_m} < 1+\epsilon. \quad \dots \quad (2)$$

Taking $\epsilon < 1$, we have $|P_{m+p}| > (1-\epsilon)|P_m|$, so that $|P_n| > a$ fixed positive number when $n > m$. So P_n cannot tend to zero.

Also by (2), P_{m+p} and P_m , are of the same sign and therefore

$$|P_{m+p}-P_m| = |P_{m+p}| - |P_m| < \epsilon |P_m|.$$

Hence $|P_{m+p}-P_m| < \epsilon'$ where ϵ' is positive and arbitrarily small. Hence P_n tends to a limit different from zero and the product P is convergent.

21'33. Absolute convergence.

The infinite product $P=(1+u_1)(1+u_2)(1+u_3)\dots$ is said to be *absolutely* convergent when $P'=(1+|u_1|)(1+|u_2|)(1+|u_3|)\dots$ is convergent. Hence $\prod (1+u_n)$ is absolutely convergent if, and only if, the series $\sum u_n$ is absolutely convergent. It may happen that $\prod (1+u_n)$ is convergent whilst the series $\sum u_n$ is divergent; in this case $\prod (1+u_n)$ is said to be *conditionally convergent* or *semi-convergent*.

21'34. Theorem. An absolutely convergent infinite product is convergent,

Let P and P' be defined as above and suppose that P' is convergent. Hence we can find m (Art. 21'31) so that

$$|(1 + |u_{m+1}|)(1 + |u_{m+2}|) \cdots (1 + |u_{m+p}|) - 1| < \epsilon$$

for every positive integer p .

$$\begin{aligned} \text{Now } & |(1 + u_{m+1})(1 + u_{m+2}) \cdots (1 + u_{m+p}) - 1| \\ & \leq |(1 + |u_{m+1}|)(1 + |u_{m+2}|) \cdots (1 + |u_{m+p}|) - 1|. \end{aligned}$$

This follows on expanding each and remembering that for any numbers $a, b, c, \dots$

$$|a + b + c + \cdots| \leq |a| + |b| + |c| + \cdots,$$

$\therefore |(1 + u_{m+1})(1 + u_{m+2}) \cdots (1 + u_{m+p}) - 1| < \epsilon$ for every positive integer p . Hence by Art. 21'32, P is convergent.

21'35. Sin x as an infinite product.

We have

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \sin \frac{x}{2} \sin \frac{x+\pi}{2} \cdots \quad (1)$$

Apply the formula (1) to each sine on the right-hand side and express it as the product of two sines; continue repeating the process. We get successively

$$\begin{aligned} \sin x &= 2^3 \sin \frac{x}{2^2} \sin \frac{x+\pi}{2^2} \sin \frac{x+2\pi}{2^2} \sin \frac{x+3\pi}{2^2} \\ &= 2^7 \sin \frac{x}{2^3} \sin \frac{x+\pi}{2^3} \sin \frac{x+2\pi}{2^3} \cdots \sin \frac{x+7\pi}{2^3}, \\ &\quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

We get finally

$$\sin x = 2^{p-1} \sin \frac{x}{p} \sin \frac{x+\pi}{p} \sin \frac{x+2\pi}{p} \cdots \sin \frac{x+(p-1)\pi}{p}, \quad \dots (2)$$

where $p = 2^n$.

Leaving the first factor as it is, combine the second factor with the last, the third with the last but one, and so on.

$$\begin{aligned} \text{As } \sin \frac{x+r\pi}{p} \sin \frac{x+(p-r)\pi}{p} &= \sin \frac{r\pi+x}{p} \sin \frac{r\pi-x}{p} \\ &= \sin^2 \frac{r\pi}{p} - \sin^2 \frac{x}{p}, \end{aligned}$$

we get

$$\sin x = 2^{p-1} \sin \frac{x}{p} \left\{ \sin^2 \frac{\pi}{p} - \sin^2 \frac{x}{p} \right\} \left\{ \sin^2 \frac{2\pi}{p} - \sin^2 \frac{x}{p} \right\} \cdots \quad \dots (3)$$

The last factor which is not paired is

$$\sin \frac{x + \frac{p}{2}\pi}{p} = \sin \left(\frac{\pi}{2} + \frac{x}{p} \right) = \cos \frac{x}{p}.$$

Hence finally

$$\begin{aligned} \sin x &= 2^{p-1} \sin \frac{x}{p} \left\{ \sin^2 \frac{\pi}{p} - \sin^2 \frac{x}{p} \right\} \left\{ \sin^2 \frac{2\pi}{p} - \sin^2 \frac{x}{p} \right\} \dots \\ &\quad \left\{ \sin^2 \frac{(p-1)\pi}{p} - \sin^2 \frac{x}{p} \right\} \cos \frac{x}{p} \end{aligned} \quad \dots (4)$$

Divide both sides of (4) by $\sin \frac{x}{p}$ and make x tend to zero.

$$\text{Since } \frac{\sin x}{\sin \frac{x}{p}} = \left(p \cdot \frac{\sin x}{x} \cdot \frac{\frac{x}{p}}{\sin \frac{x}{p}} \right) \rightarrow p \text{ as } x \rightarrow 0, \text{ we get from (4)}$$

$$p = 2^{p-1} \sin^2 \frac{\pi}{p} \sin^2 \frac{2\pi}{p} \sin^2 \frac{3\pi}{p} \dots \sin^2 \frac{(p-1)\pi}{p} \quad \dots (5)$$

Dividing (4) by (5), we have

$$\begin{aligned} \sin x &= p \sin \frac{x}{p} \left\{ 1 - \frac{\sin^2 \frac{x}{p}}{\sin^2 \frac{\pi}{p}} \right\} \left\{ 1 - \frac{\sin^2 \frac{x}{p}}{\sin^2 \frac{2\pi}{p}} \right\} \left\{ 1 - \frac{\sin^2 \frac{x}{p}}{\sin^2 \frac{3\pi}{p}} \right\} \dots \\ &\quad \dots \left\{ 1 - \frac{\sin^2 \frac{x}{p}}{\sin^2 \frac{(p-1)\pi}{p}} \right\} \cos \frac{x}{p}. \end{aligned} \quad \dots (6)$$

Now make p indefinitely great. Since

$$\left\{ p \sin \frac{x}{p} \right\} = \left\{ \frac{\sin \frac{x}{p}}{\frac{x}{p}} \cdot x \right\} \rightarrow x \text{ when } p \rightarrow \infty.$$

$$\left\{ \frac{\sin^2 \frac{x}{p}}{\sin^2 \frac{r\pi}{p}} \right\} = \left\{ \frac{\sin^2 \frac{x}{p}}{\frac{x^2}{p^2}} \cdot \frac{\frac{r^2 \pi^2}{p^2}}{\sin^2 \frac{r\pi}{p}} \cdot \frac{x^2}{r^2 \pi^2} \right\} \rightarrow \frac{x^2}{r^2 \pi^2} \text{ as } p \rightarrow \infty.$$

We finally have

$$\sin x = x \left(1 - \frac{x^2}{\pi^2} \right) \left(1 - \frac{x^2}{2^2 \pi^2} \right) \left(1 - \frac{x^2}{3^2 \pi^2} \right) \dots \text{ad inf.}$$

This result may be written in the form

$$\sin x = x \prod_{r=1}^{\infty} \left(1 - \frac{x^2}{r^2 \pi^2} \right).$$

21'36. $\cos x$ as an infinite product.

As in Art. 21'35, we first establish

$$\sin x = 2^{p-1} \sin \frac{x}{p} \sin \frac{\pi+x}{p} \sin \frac{2\pi+x}{p} \dots \sin \frac{(p-1)\pi+x}{p}. \quad \dots(1)$$

Putting $\frac{\pi}{2} + x$ for x in the above relation, we get

$$\cos x = 2^{p-1} \sin \frac{\pi+2x}{2p} \sin \frac{3\pi+2x}{2p} \dots \sin \frac{(2p-1)\pi+2x}{2p}.$$

We now combine the factors in pairs, the first with the last, the second with the last but one and so on.

The r th factor from the beginning is $\sin \frac{(2r-1)\pi+2x}{2p}$ and the r th factor from the end is $\sin \frac{(2p-2r+1)\pi+2x}{2p}$; and

$$\begin{aligned} & \sin \frac{(2r-1)\pi+2x}{2p} \cdot \sin \frac{(2p-2r+1)\pi+2x}{2p} \\ &= \sin \frac{(2r-1)\pi+2x}{2p} \cdot \sin \left\{ \pi - \frac{(2r-1)\pi-2x}{2p} \right\} \\ &= \sin \frac{(2r-1)\pi+2x}{2p} \cdot \sin \frac{(2r-1)\pi-2x}{2p} \\ &= \sin^2 \frac{(2r-1)\pi}{2p} - \sin^2 \frac{2x}{2p}, \end{aligned}$$

$$\text{we get } \cos x = 2^{p-1} \left\{ \sin^2 \frac{\pi}{2p} - \sin^2 \frac{2x}{2p} \right\} \left\{ \sin^2 \frac{3\pi}{2p} - \sin^2 \frac{2x}{2p} \right\} \dots \dots (2)$$

On making x tend to 0, we have

$$1 = 2^{p-1} \sin^2 \frac{\pi}{2p} \sin^2 \frac{3\pi}{2p} \sin^2 \frac{5\pi}{2p} \dots \dots (3)$$

Divide (2) by (3) and make p indefinitely great. Now as

$$\frac{\sin^2 \frac{2x}{2p}}{\sin^2 \frac{(2r-1)\pi}{2p}} = \left\{ \frac{\sin \frac{2x}{2p}}{\frac{2x}{2p}} \cdot \frac{(2r-1)\pi}{2p} \cdot \frac{2x}{(2r-1)\pi} \right\}^2 \rightarrow \frac{4x^2}{(2r-1)^2 \pi^2}$$

we finally get

as $p \rightarrow \infty$;

$$\cos x = \left\{1 - \frac{4x^2}{\pi^2}\right\} \left\{1 - \frac{4x^2}{3^2\pi^2}\right\} \left\{1 - \frac{4x^2}{5^2\pi^2}\right\} \cdots \text{ad inf.}$$

$$= \prod_{r=1}^{\infty} \left\{1 - \frac{4x^2}{(2r-1)^2 \pi^2}\right\}.$$

Note 1. Since $\cos x = \frac{\sin 2x}{2 \sin x}$, the expression for $\cos x$ as an infinite product may be obtained from the corresponding expressions for $\sin 2x$ and $\sin x$.

Note 2. The methods used in Arts 21'35 and 21'36 are not strictly rigorous.

21'37. Convergency of the infinite products for $\sin x$ and $\cos x$.

(i) $\sin x = x \prod \left(1 - \frac{x^2}{r^2\pi^2}\right)$ is absolutely convergent since

$$\sum \left| -\frac{x^2}{r^2\pi^2} \right| = \sum \left(\frac{x^2}{r^2\pi^2} \right)$$

or $\frac{x^2}{\pi^2} \sum \frac{1}{r^2}$ is convergent.

(ii) $\cos x = \prod \left\{1 - \frac{4x^2}{(2r-1)^2 \pi^2}\right\}$ is also absolutely convergent since

$$\frac{4x^2}{\pi^2} \sum \frac{1}{(2r-1)^2} \text{ is convergent.}$$

If however $\sin x$ and $\cos x$ be written as

$$\sin x = x \left(1 + \frac{x}{\pi}\right) \left(1 - \frac{x}{\pi}\right) \left(1 + \frac{x}{2\pi}\right) \left(1 - \frac{x}{2\pi}\right) \cdots,$$

$$\text{and } \cos x = \left(1 + \frac{2x}{\pi}\right) \left(1 - \frac{2x}{\pi}\right) \left(1 + \frac{2x}{3\pi}\right) \left(1 - \frac{2x}{3\pi}\right) \cdots,$$

the products are semi-convergent because the series $\sum_r \frac{1}{r}$ and

$$\sum \frac{1}{2r-1} \text{ are divergent.}$$

21'38. We close this chapter with the statements only of two tests for convergence, one due to Abel and the other to Dirichlet.

Abel's test: If the series $\sum u_n$ be convergent and $\{a_n\}$ be a bounded monotonic sequence of 'positive numbers, then the series $\sum a_n u_n$ is also convergent.

Dirichlet's test: If $\sum u_n$ is a series which either converges or oscillates finitely and if $\{a_n\}$ is a monotone decreasing sequence of

positive numbers which tends to the limit zero, then the series $\sum a_n u_n$ is convergent.

Examples :

[In what follows, the word convergence will mean the quality of a series as regards convergency or divergency ; and the symbols C, D, O_1 and O_2 will respectively mean convergency, divergency, finite oscillation and infinite oscillation.]

1. Test for convergence the following series :

$$(i) \quad 1 + \frac{2^2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \dots; \quad (C. H. 1966, '65)$$

$$(ii) \quad \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \dots; \quad (C. H. 1964, '61)$$

$$(iii) \quad \frac{2}{1 \cdot 4 \cdot 5} + \frac{3}{2 \cdot 5 \cdot 6} + \frac{4}{3 \cdot 6 \cdot 7} + \dots; \quad (C. H. 1962)$$

$$(iv) \quad \frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \dots; \quad (C. H. 1961)$$

$$(v) \quad \left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots.$$

[(i), (ii) - D ; (iii), (iv), (v) - C . In (v) apply Cauchy's root-test.]

2. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^s}$ where $|x| = 1$ and $s \leq 1$.

[$x=1$: D ; $x=-1$ and $s=1$: C ; $x=-1$ and $s < 1$: O_2] (C. H. 1966)

3. Test for convergence the series for which

$$(i) \quad u_{n-1} = \frac{1}{\log n}. \quad (C. H. 1960)$$

$$(ii) \quad u_n = \sqrt{n^2 + 1} - n. \quad (C. H. 1958)$$

$$(iii) \quad u_n = \frac{n+1}{n^2}. \quad (C. H. 1958)$$

$$(iv) \quad u_n = \frac{1}{(n+1 + \cos n\pi)^2}. \quad (C. H. 1955)$$

$$(v) \quad u_n = \frac{(1+\alpha)(2+\alpha) \dots (n+\alpha)}{(1+\beta)(2+\beta) \dots (n+\beta)} \quad (\beta - \alpha \neq 1). \quad (C. P. 1965)$$

[(i), (ii) D ; (iii) C if $p > 2$, D otherwise; (iv) C , sum $= \frac{\pi^2}{6} - \frac{1}{4}$; (v) C if $\beta - \alpha > 1$, D if $\beta - \alpha < 1$.]

4. Test for convergence the series :

$$(i) \frac{1}{2^{a+x}} + \frac{2^a}{3^{a+x}} + \frac{3^a}{4^{a+x}} + \dots; \quad (C. H. 1963)$$

$$(ii) 1 + \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \dots; \quad (C. H. 1953)$$

$$(iii) 1 + \frac{1}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \dots; \quad (C. H. 1957)$$

$$(iv) \frac{a}{b} + \frac{a(a+1)}{b(b+1)}x + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^2 + \dots (a, b > 0); \quad (C. H. 1956)$$

$$(v) \frac{a+x}{1} + \frac{(a+x)^2}{2!} + \frac{(a+x)^3}{3!} + \dots;$$

$$(vi) x^2(\log 2)^p + x^3(\log 3)^p + x^4(\log 4)^p + \dots;$$

$$(vii) 1 + 2x + \frac{3^2 x^2}{2!} + \frac{4^3 x^3}{3!} + \frac{5^4 x^4}{4!} + \dots (x > 0); \quad (C. H. 1961)$$

$$(viii) x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \dots; \quad (C. H. 1965)$$

$$(ix) x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \dots. \quad (C. H. 1955, 1963)$$

[(i) C if $x > 1$, D if $x \geq 1$; (ii) C if $x \leq \frac{1}{e}$, D if $x > \frac{1}{e}$; (iii) C if $x < e$, D if $x \geq e$; (iv) $(x < 1)C$, $(x > 1)D$, $(x=1, b-a > 1)C$, $(x=1, b-a \leq 1)D$. (v) compare with $\Sigma \frac{(nx)^n}{n!}$; $(x < \frac{1}{e})C$, $(x \geq \frac{1}{e})D$; (vi) $(x < 1)C$, $(x \geq 1)D$, for all p ; (vii) $(x < \frac{1}{e})C$, $(x \geq \frac{1}{e})D$; (viii) Same as (vii); (ix) $(x \leq 1)C$, $(x > 1)D$.]

5. Find whether the series for which

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot \dots \cdot (2n-2)^2}{3 \cdot 4 \cdot 5 \cdot \dots \cdot (2n-1)(2n)} x^{2n}$$

is convergent or divergent.

(C. H. 957)

[C if $-1 < x < 1$; D otherwise.]

$$6. \text{ If } \frac{u_n}{u_{n+1}} = \frac{n^k + An^{k-1} + Bn^{k-2} + Cn^{k-3} + \dots}{n^k + an^{k-1} + bn^{k-2} + cn^{k-3} + \dots},$$

where k is a positive integer, show that Σu_n is convergent if $A-a > 1$, and divergent if $A-a \leq 1$.

7. Test for convergence the hypergeometric series

$$1 + \frac{\alpha \beta}{\gamma \delta} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\delta(\delta+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\delta(\delta+1)(\delta+2)} x^3 + \dots$$

[$(x < 1), C$; $(x > 1), D$; for $x=1, (\alpha+\beta-\gamma-\delta+1 < 0), C$; $(\alpha+\beta-\gamma-\delta+1 \geq 0), D.$]

8. If Σu_n is a convergent positive series, prove that Σu_n^2 is also convergent. Give an example to show that the converse is not true.

[$u_n < 1$ for $n > m$ as $u_n \rightarrow 0$. $\therefore u_n^2 < u_n$ for $n > m$]

9. If Σu_n is a convergent positive series, show that $\Sigma \frac{u_n}{n}$ is also convergent. (C. H. 1965)

$$\left[\text{For } \frac{u_n}{n} \leq \frac{1}{2} \left(u_n^2 + \frac{1}{n^2} \right) \right]$$

10. If $u_n > 0$ and Σu_n is convergent, show that $\Sigma \frac{u_n}{1+u_n}$ is convergent.

11. If $\{u_n\}$ is a decreasing sequence of positive terms tending to the limit zero, prove that

$$\left\{ \frac{u_1 + u_2 + \dots + u_n}{n} \right\} \text{ and } \left\{ \frac{u_1 + u_3 + \dots + u_{2n-1}}{2n-1} \right\}$$

are also decreasing sequences of positive terms tending to the limit zero. Hence show that

$$(i) \quad u_1 - \frac{1}{2}(u_1 + u_2) + \frac{1}{3}(u_1 + u_2 + u_3) - \dots, \quad (C. H. 1964)$$

$$(ii) \quad u_1 - \frac{1}{3}(u_1 + u_3) + \frac{1}{5}(u_1 + u_3 + u_5) - \dots$$

are convergent.

In particular, the series

$$1 - \frac{1}{2} \left(1 + \frac{1}{2} \right) + \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \right) - \dots, \quad 1 - \frac{1}{3} \left(1 + \frac{1}{3} \right) + \frac{1}{5} \left(1 + \frac{1}{3} + \frac{1}{5} \right) - \dots$$

are convergent.

[Apply Leibnitz's test for alternating series.]

12. If $\{u_n\}$ is a decreasing sequence of positive terms and $\text{Lt } u_n = 0$, prove that $(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \dots$ is convergent.

13. If $u_1 > u_2 > \dots > u_n > \dots > 0$, then $u_1 - u_2 + u_3 - u_4 + \dots$ converges or oscillates according as $\text{Lt } u_n = 0$ or $\neq 0$.

14. Given that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ to $\infty = \log 2$, show that

$$(i) \quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots \text{ (two positive terms followed by a negative term)} = \frac{3}{2} \log 2.$$

(C. H. 1963 '65)

(ii) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots$ (one positive term followed by two negative terms) $= \frac{1}{2} \log 2$.

$$\left[\begin{aligned} (i) \text{ We have } \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots \\ \frac{1}{2} \log 2 &= 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \dots \end{aligned} \right.$$

Adding we get $\frac{3}{2} \log 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$

(ii) Let $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ to n terms ;

$$\sigma_n = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots \text{ to } n \text{ terms,}$$

$$\begin{aligned} \text{then } \sigma_{3n} &= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \dots \text{ to } n \text{ terms} \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \dots \text{ to } n \text{ terms,} \\ &= \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{ to } 2n \text{ terms}\right) = \frac{1}{2} S_{2n}. \end{aligned}$$

$$\therefore \text{Lt } \sigma_{3n} = \frac{1}{2} \text{Lt } S_{2n} = \frac{1}{2} \log 2.$$

Also $\text{Lt } \sigma_{3n+2} = \text{Lt } \sigma_{3n+1} = \text{Lt } \sigma_{3n}.$]

15. Test for convergence the series

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \dots \quad (C. H. 1963)$$

$$\left[\begin{aligned} \text{Let } \sigma_n &= 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \text{ to } n \text{ terms, then obviously} \\ \sigma_{3n} &> \left(\frac{1}{3} + \frac{1}{3} - \frac{1}{3}\right) + \left(\frac{1}{6} + \frac{1}{6} - \frac{1}{6}\right) + \dots \text{ to } n \text{ terms,} \\ &> \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right). \end{aligned} \right.$$

Now $\sum \frac{1}{n}$ is divergent, therefore $\text{Lt } \sigma_{3n}$ and consequently σ_{3n+1} and σ_{3n+2} all tend to ∞ . Hence the given series is divergent.]

16. Prove that $a^1 + a^{1+\frac{1}{2}} + a^{1+\frac{1}{2}+\frac{1}{3}} + \dots$ is convergent or divergent according as $a < \frac{1}{e}$ or $\geq \frac{1}{e}$.

17. Apply the general principle of convergence to discuss the convergence of

$$(i) \sum \frac{(-1)^{n-1}}{n}, \quad (ii) \sum \frac{1}{n(n+1)} \quad (C. H. 1960), \quad (iii) \sum \frac{1}{n}.$$

[(i) Let $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ to n terms. Then

$$|S_{n+p} - S_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \dots + \frac{(-1)^{p+1}}{n+p} \right|.$$

If p be odd, $|S_{n+p} - S_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \dots + \frac{1}{n+p} \right|$
 $= \left| \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3} \right) - \dots - \left(\frac{1}{n+p-1} - \frac{1}{n+p} \right) \right|$
 $< \frac{1}{n+1}.$

If p be even

$$|S_{n+p} - S_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \dots - \frac{1}{n+p} \right|$$

$$= \left| \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3} \right) - \dots - \left(\frac{1}{n+p-2} - \frac{1}{n+p-1} \right) - \frac{1}{n+p} \right|$$

$$< \frac{1}{n+1}.$$

Thus in both cases

$$|S_{n+p} - S_n| < \frac{1}{n+1} < \epsilon, \text{ if } n > \left(\frac{1}{\epsilon} - 1 \right).$$

Hence we can find m such that for $n > m$ and $p > 0$,
 $|S_{n+p} - S_n| < \epsilon$

$$\therefore \sum \frac{(-1)^{n-1}}{n} \text{ is convergent.}$$

(ii) Let $S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$ to n terms.

$$|S_{n+p} - S_n| = \left| \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \dots + \frac{1}{(n+p)(n+p+1)} \right|$$

$$= \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} + \dots + \frac{1}{n+p} - \frac{1}{n+p+1} \right|$$

$$= \left| \frac{1}{n+1} - \frac{1}{n+p+1} \right| < \frac{1}{n+1} < \epsilon,$$

if $n > \left(\frac{1}{\epsilon} - 1 \right)$. Hence as in (i), $\sum \frac{1}{n(n+1)}$ is convergent.

(iii) Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Suppose that $\sum \frac{1}{n}$ is convergent. Then we find m such that for $n \geq m$ and $p > 0$,
 $|S_{n+p} - S_n| < \epsilon$.

$$\text{Now } |S_{n+p} - S_n| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} \right| < \epsilon.$$

Also taking $n=m$ and $p=m$, we obtain

$$\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m} > \frac{m}{2m} > \frac{1}{2}.$$

Thus $|S_{m+m} - S_m| > \frac{1}{2}$ and cannot be less than ϵ .

This contradiction shows that $\sum \frac{1}{n}$ is not convergent and as it is a positive series, it must diverge to ∞ .]

18. If $f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$, prove that

$$f(z) \times f(z') = f(z+z'). \quad (C. H. 1962)$$

[We have

$$f(az) = 1 + az + \frac{a^2 z^2}{2!} + \dots + \frac{a^n z^n}{n!} + \dots$$

$$f(az') = 1 + az' + \frac{a^2 z'^2}{2!} + \dots + \frac{a^n z'^n}{n!} + \dots$$

As the series on the right side are absolutely convergent,

$$\begin{aligned} f(az) \times f(az') &= \left(1 + az + \frac{a^2 z^2}{2!} + \dots + \frac{a^n z^n}{n!} + \dots \right) \\ &\quad \times \left(1 + az' + \frac{a^2 z'^2}{2!} + \dots + \frac{a^n z'^n}{n!} + \dots \right) \end{aligned}$$

The co-efficient of a^n in the product

$$\begin{aligned} &= \frac{z^n}{n!} + \frac{z^{n-1} z'}{(n-1)! 1!} + \dots + \frac{z z'^{n-1}}{1! (n-1)!} + \frac{z'^n}{n!} \\ &= \frac{1}{n!} (z^n + {}^n C_1 z^{n-1} z' + \dots + {}^n C_{n-1} z z'^{n-1} + {}^n C_n z'^n) \\ &= \frac{1}{n!} (z+z')^n. \end{aligned}$$

Hence $f(az) \times f(az')$

$$\begin{aligned} &= 1 + a(z+z') + \frac{a^2 (z+z')^2}{2!} + \dots + \frac{a^n (z+z')^n}{n!} + \dots \\ &= f\{a(z+z')\}. \end{aligned}$$

On putting $a=1$, we get

$$f(z) \times f(z') = f(z+z').]$$

19. To discuss the convergence of the Binomial series

$$1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots$$

i.e., the expansion of $(1+x)^m$.

(C. H. 1959)

[In the discussion, we shall use Ex. 7, viz., the hypergeometric series $1 + \frac{\alpha\beta}{\gamma\delta} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\delta(\delta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\delta(\delta+1)(\delta+2)} + \dots$

is convergent if $\gamma + \delta - \alpha - \beta > 1$ and divergent if $\gamma + \delta - \alpha - \beta \leq 1$.

In the first place we note that when m is a positive integer the Binomial series is a finite series; in particular when $m=0$, $(1+x)^m = 1$.

When m is not a positive integer, $\frac{u_{n+1}}{u_n} = \frac{m-n+1}{n} \cdot x$;

$$\therefore \text{Lt } \left| \frac{u_{n+1}}{u_n} \right| = \text{Lt } \left| \frac{m-n+1}{n} \cdot x \right| = |x|;$$

$\therefore$ (i) if $|x| < 1$, the series is convergent,

and (ii) if $|x| > 1$, the series is divergent.

Thus whatever m may be, $(1+x)^m$ can be expanded in a convergent series only when $|x| < 1$.

(iii) When $x=1$, the series is

$$1 + \frac{m}{1} + \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)}{3!} + \dots$$

Here again we need consider only positive fractional values and negative values of m .

Case I. m is a positive fraction.

Let $m-r$ be the first negative factor; we can then write the series as

$$1 + \frac{m}{1} + \dots + \frac{m(m-1)\dots(m-r+1)}{r!} \times \left\{ 1 + \frac{m-r}{r+1} + \frac{(m-r)(m-r-1)}{(r+1)(r+2)} + \dots \right\} \quad \dots \quad (1)$$

Let us now consider the series!

$$1 + \frac{m-r}{r+1} + \frac{(m-r)(m-r-1)}{(r+1)(r+2)} + \dots \quad \dots \quad (2)$$

in which the terms are alternately positive and negative. From this we derive the series

$$1 + \frac{r-m}{r+1} + \frac{(r-m)(r-m+1)}{(r+1)(r+2)} + \dots, \quad \dots \quad (3).$$

in which the terms are all positive. If (3) be convergent, then (2) and therefore (1) will be absolutely convergent. Now (3) is a hypergeometric series in which $\alpha = r-m$, $\beta = r$, $\delta = r+1$.

For convergency

$$r + \delta - \alpha - \beta > 1, \text{ i.e., } r+1-r+m > 1, \text{ or } m > 0$$

which is so, since m is a positive fraction. Thus in this case, the Binomial series is absolutely convergent.

Case II. When m is a negative number $= -\mu$ say where $\mu > 0$.

$$\begin{aligned} \text{Then } 1 + \frac{m}{1} + \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)}{3!} + \dots, \\ = 1 - \frac{\mu}{1} + \frac{\mu(\mu+1)}{2!} - \frac{\mu(\mu+1)(\mu+2)}{3!} + \dots, \quad \dots \quad (4) \end{aligned}$$

From this we derive the positive series

$$1 + \frac{\mu}{1} + \frac{\mu(\mu+1)}{2!} + \frac{\mu(\mu+1)(\mu+2)}{3!} + \dots. \quad \dots \quad (5)$$

This is a hypergeometric series in which $\alpha = \mu$, $\beta = r$, $\delta = 1$.

$\therefore r + \delta - \alpha - \beta = 1 - \mu < 1$ as $\mu > 0$. So (5) is divergent and therefore (4) cannot be absolutely convergent.

The series (4) may be (i) semi-convergent, or (ii) divergent (including oscillatory).

Writing the series (4) as $u_0 + u_1 + u_2 + \dots$, we have

$$\frac{u_{n+1}}{u_n} = -\frac{\mu+n}{1+n} \quad \text{or} \quad \left| \frac{u_{n+1}}{u_n} \right| = \frac{\mu+n}{1+n}.$$

If $\mu > 1$, $|u_{n+1}| > |u_n|$; the terms increase numerically as n increases and the series cannot be convergent. If $\mu < 1$, the terms decrease numerically as n increases and will be convergent if $\text{Lt } u_n = 0$.

$$\text{Now } \left| \frac{u_{n+1}}{u_0} \right| = \left| \frac{u_{n+1}}{u_n} \cdot \frac{u_n}{u_{n-1}} \cdot \dots \cdot \frac{u_2}{u_1} \cdot \frac{u_1}{u_0} \right| \text{ and as } u_0 = 1,$$

$$\log |u_{n+1}| = \sum \log \left| \frac{u_{n+1}}{u_n} \right| = \sum \log \frac{\mu+n}{n+1} = \sum \log \left\{ 1 + \frac{\mu-1}{n+1} \right\}.$$

Since $\lim_{x \rightarrow \infty} \frac{\log \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \log \left(1 + \frac{1}{x}\right)^x = \log e = 1,$

it follows that

$$\lim_{n \rightarrow \infty} \frac{\log \left(1 + \frac{\mu-1}{n+1}\right)}{\frac{\mu-1}{n+1}} = 1.$$

Now consider the two series $\sum \log \left(1 + \frac{\mu-1}{n+1}\right)$ and $\sum \frac{\mu-1}{n+1}$; the ratio of their corresponding terms is 1. Also $\sum \frac{\mu-1}{n+1}$ is a divergent series and therefore $\sum \frac{\mu-1}{n+1} = -\infty$ if $\mu < 1$.

$$\therefore \sum \log \left(1 + \frac{\mu-1}{n+1}\right) = -\infty \text{ if } \mu < 1,$$

i.e., $\lim_{n \rightarrow \infty} \log |u_{n+1}| = -\infty$ if $\mu < 1$, or $\lim_{n \rightarrow \infty} |u_{n+1}| = 0$ if $\mu < 1$. Hence it follows from Leibnitz's test that if $\mu < 1$ or $0 > m > -1$, the series is semi-convergent.

If $\mu = 1$, the series is $1 - 1 + 1 - 1 + \dots$ which oscillates finitely.

If $\mu > 1$ or $m < -1$, we have seen that the terms increase numerically beyond limit and as the terms are alternately of opposite signs, the series oscillates between infinite limits.]

20. Test for convergence the expansions of (i) e^x , (ii) $\log(1+x)$.
(C. H. 1959)

[(i) C for all x ; (ii) C when $|x| < 1$, for $x = 1$, the series is semi-convergent; for $x = -1$, the series diverges.]

21. Show that the limit when $n \rightarrow \infty$ of

$$P_n = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \dots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n}$$

is finite.

[P_n consists of $2n$ factors; taking consecutive factors in pairs,

$$P_n = \prod_{i=1}^n \left[\frac{(2n)^2 - 1}{(2n)^2} \right] = \prod_{i=1}^n \left(1 - \frac{1}{4n^2} \right).$$

Since $\sum \left| -\frac{1}{4n^2} \right| = \sum \frac{1}{4n^2}$ is convergent, P is absolutely convergent and therefore finite.]

22. Show that the limit when $n \rightarrow \infty$ of

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdots \frac{2n-2}{2n-3} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n}{2n-1}$$

is finite.

$$\left[\text{Here } 2P_n = 2 \prod_2^n \left[1 - \frac{1}{(2n-1)^2} \right] \right]$$

23. Prove that

$$(i) \left(1 + \frac{1}{2} \right) \left(1 + \frac{1}{3} \right) \left(1 + \frac{1}{4} \right) \cdots \rightarrow \infty.$$

$$\left[\text{Here } P_n = \prod_2^n \left(1 + \frac{1}{n} \right) = \frac{1}{2} (n+1) \rightarrow \infty. \right]$$

$$(ii) \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{3} \right) \left(1 - \frac{1}{4} \right) \cdots \rightarrow 0.$$

$$\left[\text{Here } P_n = \prod_2^n \left(1 - \frac{1}{n} \right) = \frac{1}{n} \rightarrow 0. \right]$$

$$(iii) \left(1 - \frac{1}{2^2} \right) \left(1 - \frac{1}{3^2} \right) \left(1 - \frac{1}{4^2} \right) \cdots \rightarrow \frac{1}{2}.$$

$$\left[\text{Here } P_n = \prod_2^n \left[\frac{n-1}{n} \cdot \frac{n+1}{n} \right] = \prod_2^n \left(\frac{1}{2} \cdot \frac{n+1}{n} \right) \rightarrow \frac{1}{2}. \right]$$

24. The product

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{n}{n-1} \cdot \frac{n+1}{n} \cdots$$

is not convergent although $u_n = \frac{n+1}{n} \rightarrow 1$ as $n \rightarrow \infty$.

$$\left[\text{Here } P_n = \prod_1^n u_n = n+1 \rightarrow \infty \text{ as } n \rightarrow \infty. \right]$$

25. For what values of x has the infinite product

$$(1+a)(1+ax)(1+ax^2)(1+ax^3) \cdots$$

a finite value?

$$[|x| < 1]$$

26. Show that

$$\frac{\sin x}{x} = \cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \cdots \text{to } \infty.$$

(Euler's product)

27. Prove that

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \dots \text{ad inf.} \quad (\text{Wallis's theorem})$$

Deduce that

$$\sqrt{\pi} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \cdot \frac{2.4.6. \dots (2n)}{1.3.5. \dots (2n-1)}.$$

[In the expression for $\sin x$ in factors (Art. 21'35), put $x = \frac{\pi}{2}$.]

28. Show that

$$(i) \sinh x = x \left(1 + \frac{x^2}{\pi^2} \right) \left(1 + \frac{x^2}{2^2 \pi^2} \right) \left(1 + \frac{x^2}{3^2 \pi^2} \right) \dots$$

$$= x \prod_1^{\infty} \left(1 + \frac{x^2}{r^2 \pi^2} \right),$$

$$(ii) \cosh x = \left(1 + \frac{4x^2}{\pi^2} \right) \left(1 + \frac{4x^2}{3^2 \pi^2} \right) \left(1 + \frac{4x^2}{5^2 \pi^2} \right) \dots$$

$$= \prod_1^{\infty} \left\{ 1 + \frac{4x^2}{(2r-1)^2 \pi^2} \right\}.$$

[Put ix for x in the expressions of $\sin x$ and $\cos x$ in factors.]

29. Prove that

$$(i) \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6},$$

$$(ii) \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}.$$

$$\left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = x \left(1 - \frac{x^2}{\pi^2} \right) \left(1 - \frac{x^2}{2^2 \pi^2} \right) \left(1 - \frac{x^2}{3^2 \pi^2} \right) \dots \right]$$

Take logarithms, equate the coefficients of x^2 and x^4 , after expansion.

For (i), see also Ex. 1. Art. 15'8.]

30. Prove that

$$(i) \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8},$$

$$(ii) \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96},$$

$$(iii) \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}.$$

[For (i) and (ii), use the expressions of $\cos x$ in series and in factors and apply the method of Ex. 29. For (i), see also Exs. 2, 5, Art. 15'8.]

Appendix

APPROXIMATE EVALUATION OF A DEFINITE INTEGRAL

1. Trapezoidal Rule.

The definite integral $\int_a^b f(x) dx$ geometrically represents the area included between the arc PQ of the curve $y=f(x)$, the ordinates AP and BQ at $x=a$ and $x=b$ respectively and the x -axis. This is approximately equal to the area of the trapezium $APQB$ which is

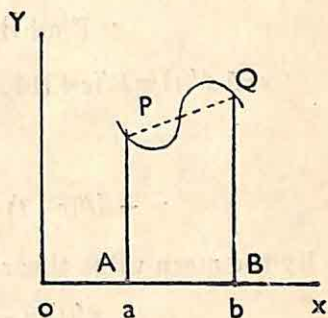


Fig. 27

$\frac{1}{2}(b-a)\{f(a)+f(b)\}$. Thus an approximate value of $\int_a^b f(x) dx$ is

$\frac{1}{2}(b-a)\{f(a)+f(b)\}$. If $AP=y_1$, $BQ=y_2$ and $b-a=h$, this is $\frac{1}{2}h(y_1+y_2)$.

If $f(x)$ has two derivatives and ξ lies in (a, b) , the exact value of $\int_a^b f(x) dx$ is given by

$$\int_a^b f(x) dx = \frac{1}{2}(b-a)\{f(a)+f(b)\} - \frac{1}{12}(b-a)^3 f''(\xi).$$

To establish this, let $a=c-h$ and $b=c+h$ and consider the function

$$\phi(t) = \psi(t) - \left(\frac{t}{h}\right)^3 \psi(h)$$

where

$$\psi(t) = \int_{c-t}^{c+t} f(x) dx - t\{f(c+t)+f(c-t)\}.$$

If $F(x)$ be the primitive of $f(x)$, so that $F'(x)=f(x)$,

$$\int_{c-t}^{c+t} f(x) dx = F(c+t) - F(c-t).$$

$$\begin{aligned}\therefore \phi(t) &= \int_{c-t}^{c+t} f(x) \, dx - t\{f(c+t) + f(c-t)\} - \left(\frac{t}{h}\right)^3 \psi(h) \\ &= F(c+t) - F(c-t) - t\{f(c+t) + f(c-t)\} - \left(\frac{t}{h}\right)^3 \psi(h) \\ \text{and } \phi'(t) &= F'(c+t) + F'(c-t) - \{f(c+t) + f(c-t)\} \\ &\quad - t\{f'(c+t) - f'(c-t)\} - \frac{3t^2}{h^3} \psi(h) \\ &= -t\{f'(c+t) - f'(c-t)\} - \frac{3t^2}{h^3} \psi(h).\end{aligned}$$

By the mean value theorem.

$$\phi'(t) = -2t^2 \left\{ f''(\xi) + \frac{3}{2h^3} \psi(h) \right\}$$

where ξ lies in $(c-t, c+t)$.

Now $\phi(0) = \phi(h) = 0$, and therefore by Rolle's theorem $\phi'(t_1) = 0$ where $0 < t_1 < h$.

$$\text{Hence } -2t_1^2 \left\{ f''(\xi) + \frac{3}{2h^3} \psi(h) \right\} = 0,$$

$$\text{which gives } \psi(h) = -\frac{2h^3}{3} f''(\xi),$$

$$\text{i.e., } \int_{c-h}^{c+h} f(x) \, dx = h\{f(c+h) + f(c-h)\} - \frac{2h^3}{3} f''(\xi)$$

$$\text{or, } \int_a^b f(x) \, dx = \frac{b-a}{2} \{f(b) + f(a)\} - \frac{1}{12} (b-a)^3 f''(\xi),$$

where ξ lies in $(c-t, c+t)$ and therefore in $(c-h, c+h)$ or, (a, b) .

The result may be expressed in terms of the primitive function $F(x)$ as

$$F(b) = F(a) + \frac{1}{2} (b-a) \{F'(a) + F'(b)\} - \frac{1}{12} (b-a)^3 F'''(\xi),$$

where $a < \xi < b$. (See example 16, Chapter 7).

The approximate formula

$$\int_a^b f(x) \, dx = \frac{1}{2} (b-a) \{f(a) + f(b)\}$$

is exact when $f(x)$ is linear, for in this case $f''(x) = 0$.

In practice, we divide AB into any number n of equal parts and apply the formula to the n trapezia thus obtained. If the ordinates at A and the successive points of section be $y_1, y_2, y_3, \dots, y_{n+1}$, we get an approximate value of

$\int_a^b f(x) dx$ in the form

$$\frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3)$$

$$+ \dots + \frac{1}{2}h(y_n + y_{n+1})$$

$$= h \left[\frac{y_1 + y_{n+1}}{2} + y_2 + y_3 + \dots + y_n \right], \text{ where } h = \frac{b-a}{n}.$$

We have thus the following rule.

To obtain an approximate value of the integral $\int_a^b f(x) dx$ divide

the interval (a, b) into any number n of equal parts. Let $y_1, y_2, \dots, y_{n+1}$ be the ordinates at a and the successive points of division. Take the average of the first and the last ordinates and add to it the sum of the other ordinates; multiply the result by the distance between two consecutive ordinates. This gives an approximate value of the integral.

2. Simpson's Rule.

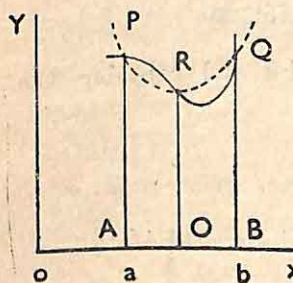


Fig. 29

Let PQ be an arc of the curve $y = f(x)$, AP, BQ the ordinates at $x=a, x=b$. Bisect AB at O and let OR be the ordinate through O . Let a parabola with its axis parallel to the y -axis be drawn through P, R, Q . Then the area included between the parabola, the ordinates AP, BQ , and the x -axis is taken to be an approximate value of the area

$$\text{represented by } \int_a^b f(x) dx.$$

The equation of the parabola is of the form $y = l + mx + nx^2$.

Take OR as the y -axis and let $AO=OB=h=\frac{b-a}{2}$. The coordinates of P, R, Q are then $(-h, y_1), (0, y_2), (h, y_3)$ where the ordinates AP, OR , and BQ are respectively y_1, y_2, y_3 . As P, R, Q , are on the parabola,

$$y_1 = l - mh + nh^2, y_2 = l, y_3 = l + mh + nh^2.$$

$$\therefore y_1 + y_3 = 2l + 2nh^2 = 2y_2 + 2nh^2, \text{ whence } 2nh^2 = y_1 + y_3 - 2y_2.$$

Now the area between the parabola PRQ , the ordinates AP, BQ and the x -axis is

$$\begin{aligned} \int_{-h}^h (l + mx + nx^2) dx &= \left[lx + \frac{1}{2} mx^2 + \frac{1}{3} nx^3 \right]_{-h}^h \\ &= 2lh + \frac{2}{3} nh^3 = h \left[2l + \frac{2}{3} nh^2 \right] = h \left[2y_2 + \frac{1}{3} (y_1 + y_3 - 2y_2) \right] \\ &= \frac{h}{3} (y_1 + y_3 + 4y_2). \end{aligned}$$

Hence an approximate value of $\int_a^b f(x) dx$ is

$$\frac{b-a}{6} \left\{ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right\}; \text{ this is known as Simpson's rule.}$$

If $f(x)$ has four derivatives and ξ lies between a and b , the exact value of the integral is given by

$$\int_a^b f(x) dx = \frac{1}{6} (b-a) \left\{ \begin{array}{l} \\ \end{array} \right\} + f(b) + 4f\left(\frac{a+b}{2}\right) \left\{ \begin{array}{l} \\ \end{array} \right\}$$

$$- \frac{1}{2880} (b-a)^5 f^{(iv)}(\xi).$$

To prove this, we write $a=c-h, b=c+h$ and consider the function

$$\phi(t) = \psi(t) - \left(\frac{t}{h}\right)^5 \psi(h),$$

where

$$\psi(t) = \int_{c-t}^{c+t} f(x) dx - \frac{1}{3} t \{f(c+t) + f(c-t) + 4f(c)\}.$$

Differentiating thrice in succession, we get

$$\phi'(t) = \frac{2}{3} \{f(c+t) - 2f(c) + f(c-t)\}$$

$$- \frac{1}{3} t \{f'(c+t) - f'(c-t)\} - \frac{5t^4}{h^5} \psi(h),$$

$$\begin{aligned}\phi''(t) &= \frac{1}{3} \{f'(c+t) - f'(c-t)\} \\ &\quad - \frac{1}{3} t \{f''(c+t) + f''(c-t)\} - \frac{20t^3}{h^5} \psi(h), \\ \phi'''(t) &= -\frac{1}{3} t \{f'''(c+t) - f'''(c-t)\} - \frac{60t^3}{h^5} \psi(h).\end{aligned}$$

By the mean value theorem

$$\phi'''(t) = -\frac{2}{3} t^2 \{f^{(iv)}(\xi) + \frac{90}{h^5} \psi(h)\},$$

where ξ lies in $(c-t, c+t)$.

Now $\phi(0) = \phi(h) = 0$, and so by Rolle's theorem $\phi'(t_1) = 0$ where $0 < t_1 < h$. Also $\phi'(0) = \phi'(t_1) = 0$ and therefore $\phi''(t_2) = 0$ where $0 < t_2 < t_1 < h$. And lastly $\phi''(0) = \phi''(t_2) = 0$ and so $\phi'''(t_3) = 0$ where $0 < t_3 < t_2 < h$.

$$\text{Hence } \phi'''(t_3) = -\frac{2}{3} t_3^2 \left\{ f^{(iv)}(\xi) + \frac{90}{h^5} \psi(h) \right\} = 0,$$

whence $\psi(h) = -\frac{h^5}{90} f^{(iv)}(\xi)$ where ξ lies in $(c-t_3, c+t_3)$ and therefore in $(c-h, c+h)$, i.e., in (a, b) . Thus

$$\int_{c-h}^{c+h} f(x) dx \times \frac{1}{3} h \left\{ f(c+h) + f(c-h) + 4f(c) \right\} = -\frac{h^5}{90} f^{(iv)}(\xi),$$

$$\begin{aligned}\text{or, } \int_a^b f(x) dx &= \frac{1}{6} (b-a) \left\{ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right\} \\ &\quad - \frac{1}{2880} (b-a)^5 f^{(iv)}(\xi).\end{aligned}$$

The approximate formula

$$\int_a^b f(x) dx = \frac{1}{6} (b-a) \left\{ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right\}$$

is exact for polynomials of the third or lower degree as $f^{(iv)}(x)$ is then equal to 0.

For a practical application of the approximate formula, we divide

(a, b) into an *even* number of equal parts, say $2n$. If $y_1, y_2, y_3, \dots, y_{2n+1}$

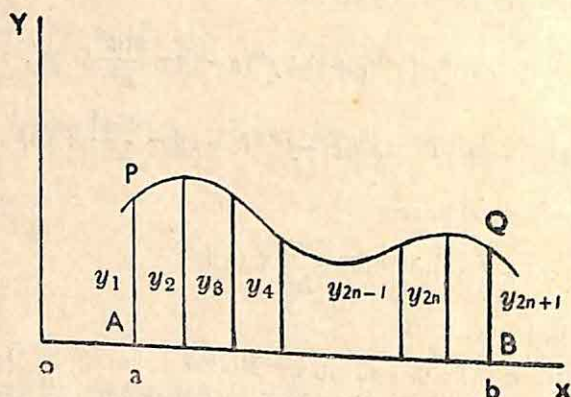


Fig. 30

be the ordinates at a and the successive points of division, and $h = \frac{b-a}{2n}$, applying the approximate formula to each of the triads

$(y_1, y_2, y_3), (y_3, y_4, y_5), \dots, (y_{2n-1}, y_{2n}, y_{2n+1})$, we get

$$\begin{aligned} & \frac{1}{3}h(y_1 + y_3 + 4y_2) + \frac{1}{3}h(y_3 + y_5 + 4y_4) + \dots \\ & \quad + \frac{1}{3}h(y_{2n-1} + y_{2n+1} + 4y_{2n}) \\ & = \frac{1}{3}h\{y_1 + y_{2n+1} + 2(y_3 + y_5 + \dots + y_{2n-1}) + 4(y_2 + y_4 + \dots + y_{2n})\} \end{aligned}$$

as an approximate value of the integral $\int_a^b f(x) dx$.

Hence the following rule :

To the sum of the first and last ordinates, add twice the sum of the remaining odd ordinates and four times the sum of all the even ordinates ; multiply the result by one-third of the distance between two consecutive ordinates. This is **Simpson's one-third Rule**.

Examples :

1. Dividing the interval $(0, 1)$ into two equal parts and applying Simpson's rule to the integral

$$\int_0^1 \frac{dx}{1+x^2},$$

show that an approximate value of π is 3.1416 .

(C. H. 1962)

Obtain a better value using $h=0.25$. [3.1416]

2. Find an approximate value of $\log 2$ by applying Simpson's rule to

$$\int_1^2 \frac{dx}{x}$$

using 11 ordinates, being given that

$$\begin{array}{ll} y_2 = f(1.1) = .90909, & y_3 = f(1.2) = .83333, \\ y_4 = f(1.3) = .76923, & y_5 = f(1.4) = .71429, \\ y_6 = f(1.5) = .66667, & y_7 = f(1.6) = .62500, \\ y_8 = f(1.7) = .58824, & y_9 = f(1.8) = .55556, \\ y_{10} = f(1.9) = .52632, & y_{11} = f(2) = .50000 \end{array}$$

where $f(x) = \frac{1}{x}$.

[$\log 2 = .69315$; the actual value is $.693147$.]

3. Given $\log 1 = 0$, $\log 2 = 0.6931$, $\log 3 = 1.0986$, $\log 4 = 1.3863$.

$\log 5 = 1.6094$, evaluate $\int_1^5 \log x \, dx$ by direct integration and also by

Simpson's rule using five ordinates. Find the error.

4. Show that $8.9 < \int_3^5 \sqrt{4+x^2} \, dx < 9$.

5. Apply Simpson's rule with five ordinates to calculate

$$\int_1^2 \sqrt{x - \frac{1}{x}} \, dx$$

to two decimal places.

[0.84]

6. Use Simpson's rule to prove that

(i) the volume of a sphere of radius r is $\frac{4}{3} \pi r^3$;

(ii) the volume of a right circular cone of height h and radius of base r is $\frac{1}{3} \pi r^2 h$.

7. If $y = f(x) = a + bx + cx^2$, $y_1 = f(p)$, $y_2 = f(p+h)$ and $y_3 = f(p+2h)$, prove that

$$\int_p^{p+2h} f(x) \, dx = \frac{1}{3} h(y_1 + y_3 + 4y_2).$$

Hence obtain an approximate value of

$$\int_0^{0.2} (1-2x^2)^{\frac{1}{3}} dx.$$

$$[\text{Take } f(x) = (1-2x^2)^{\frac{1}{3}} = 1 - \frac{2}{3}x^2 \dots$$

$$= a + bx + cx^2. \text{ Ans. } 0.1932]$$

8. A curve is drawn through the points given by the following table :

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, the x -axis, and the lines $x=1$, $x=4$. [7.78]

9. Prove that the area between the axis of x and the curve

$$y = a + bx + cx^2 + dx^3$$

from $x=0$ to $x=3h$ is

$$\frac{3}{8} h(y_0 + 3y_1 + 3y_2 + y_3),$$

where y_r is the value of y when $x=rh$.

Hence find an approximation to the value of the integral

$$\int_0^{\frac{\pi}{2}} (1+6 \sin \theta)^{\frac{1}{2}} d\theta. \quad [0.82]$$

10. If the interval (a, b) be divided into $2n$ equal parts and $y_1, y_2, \dots, y_{2n+1}$ be the ordinates to $f(x)$ at a and the successive

points of section, prove that the value of $\int_a^b f(x) dx$ lies between

$$2h(y_2 + y_4 + \dots + y_{2n})$$

and

$$h(y_1 + y_{2n+1}) + 2h(y_3 + y_5 + \dots + y_{2n-1}).$$

h being the distance between consecutive ordinates.

